

The index of lattice Dirac operators and K-theory



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Shoto Aoki(U. Tokyo), HF, Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(Osaka U.), and Satoshi Yamaguchi (Osaka U.), "The index of lattice Dirac operators and K-theory," [arXiv:2407.17708](https://arxiv.org/abs/2407.17708), [2501.02873](https://arxiv.org/abs/2501.02873)

Let me introduce myself first.

My major is particle physics, in particular, numerical simulation of lattice QCD(quantum chromo-dynamics).

During 2009-2010 I was an assistant professor **here in Nagoya U.** specially appointed for the GCOE program “Quest for Fundamental Principles in the Universe”.

At the interview, Prof. Naoshi Sugiyama asked me that

“In this GCOE program, please study something **interdisciplinary** among different fields.”

I tried but I could not write any papers, **which was my regret.**

But after >10 years, I finally found something **interdisciplinary with mathematicians** and wrote some papers. This talk on a work using K-theory is one of them.

What is the index of Dirac operators ?

$$D\psi = 0 \quad D := \gamma^\mu(\partial_\mu + iA_\mu) \quad \text{we consider U(1) or SU(N) group}$$

$$\underbrace{\text{Ind}(D)}_{n_+ - n_-} = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}F_{\rho\sigma}) = \mathbf{E} \cdot \mathbf{B}$$

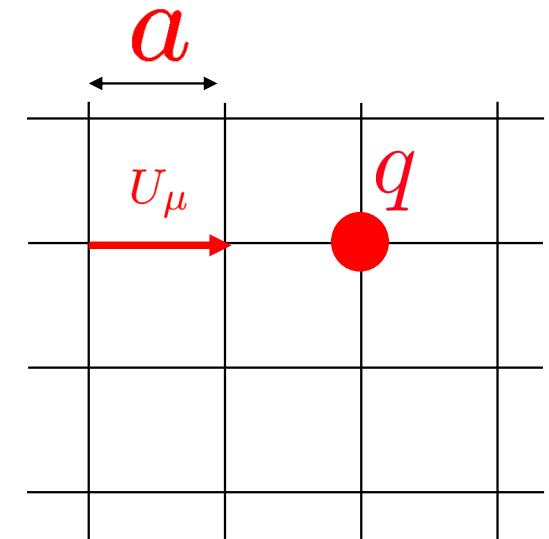
#sol with + chirality #sol with - chirality

Topological charge

Important both in physics and mathematics to understand gauge field topology, which is non-perturbative.

What is lattice gauge theory?

It is a (**non-perturbative**) regularization of quantum field theory with lattice spacing a



Gauge fields (gluons) live on links

$$U_{n,\mu} = \exp(igaA_\mu(n + \hat{\mu}/2))$$

Fermions (quarks) live on sites $q_n = q(n)$

The Lagrangian is given by for example,

$$L = \beta \sum_{\mu,\nu=1}^4 \text{Tr}[U_{n,\mu}U_{n+\mu,\nu}U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger] + \bar{q}_n \left[\sum_{\mu} \gamma_{\mu} \frac{U_{n,\mu}q_{n+\hat{\mu}} - U_{n-\hat{\mu},\mu}^\dagger q_{n-\hat{\mu}}}{2a} + m \right] q_n$$

which converges to QCD Lagrangian in the $a \rightarrow 0$ limit.

Our goal

= A mathematical formulation of the index (theorem) on a lattice.

In continuum, Dirac operator is **a differential** operator.

$$D\psi = \gamma^\mu (\partial_\mu + iA_\mu)\psi.$$

On lattice, Dirac operator is **a difference** operator.

$$D\psi = \gamma^\mu [U_\mu(x)\psi(x+\mu a) - \psi(x)]/a.$$

Mathematically nontrivial.

[Related works by mathematicians: Kubota 2020, Yamashita 2021]

Difficulty in lattice gauge theory

Both of Dirac index and topology are difficult on the lattice:

- It is difficult to define the chiral zero modes, since the standard lattice Dirac operators break the chiral symmetry.
- Lattice discretization of space time makes the topology not well-defined.

A traditional solution = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] satisfying the Ginsparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov} \gamma_5 = a D_{ov} \gamma_5 D_{ov}$$

a modified **chiral symmetry is exact** [Luescher 1998],

and the index is well-defined: $\text{Ind} D_{ov} = \text{Tr} \gamma_5 \left(1 - \frac{a D_{ov}}{2} \right)$

[Hasenfratz et al. 1998]

but this definition is so far limited to the even-dimensional flat periodic lattice.

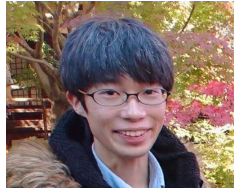
This work = an alternative mathematical formulation of the lattice Dirac index.

In our formulation,

- No exact chiral symmetry is needed : the standard **Wilson Dirac operator is good enough.**
- **K theory is used** to show equality to the continuum Dirac index.
- **Wider application than the overlap** Dirac operator to the systems with nontrivial boundaries in any dimensions.

Phys-Math collaborators

Physicists



Shoto Aoki



Tetsuya Onogi

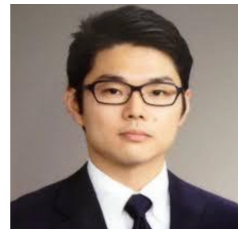


Satoshi
Yamaguchi

Mathematicians



Mikio Furuta



Shinichiroh Matsuo

Physicist-friendly Dirac index project

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Mod-two index version [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]
- Lattice version (without boundary) [Aoki, F, Furuta, Matsuo, Onogi, Yamaguchi 2024] = this work.

Q. How physicist-friendly?

A. We do not need to take care of chiral symmetry and unphysical boundary conditions in our formulation.

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7. Summary and discussion

Continuum-> Lattice : derivative -> difference

Continuum Dirac operator

$$D\psi(x) = \gamma^\mu (\partial_\mu) \psi(x) = \int dp \gamma^\mu (i p_\mu) \tilde{\psi}(p) e^{ipx}$$

(A naïve) lattice Dirac operator

$$D\psi(x) = \gamma^\mu \frac{\psi(x + \hat{\mu}a) - \psi(x - \hat{\mu}a)}{2a} = \int dp \gamma^\mu \frac{e^{ip(x+\hat{\mu}a)} - e^{ip(x-\hat{\mu}a)}}{2a} \tilde{\psi}(p)$$
$$= \int dp \gamma^\mu i \frac{\sin p_\mu a}{a} \tilde{\psi}(p) e^{ipx}.$$

a :lattice spacing

$\hat{\mu}$: unit vector in μ direction.

which has zero points at $p_\mu = 0, \frac{\pi}{a}$ (phys) Doublers appear!
(math) Ellipticity is lost!

Wilson Dirac operator

a :lattice spacing

$\hat{\mu}$: unit vector in μ direction.

The Wilson Dirac operator is commonly used in lattice gauge theory.

$$D_W = \sum_{\mu} \left[\gamma^{\mu} \frac{\nabla_{\mu}^f + \nabla_{\mu}^b}{2} - \frac{a}{2} \nabla_{\mu}^f \nabla_{\mu}^b \right]$$

$$\nabla^f \psi(x) = \frac{\psi(x + \hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^b \psi(x) = \frac{\psi(x) - \psi(x - \hat{\mu}a)}{a}$$

The additional term corresponds the Laplacian and the Fourier transformation

$$\sum_{\mu} \gamma^{\mu} i \frac{\sin p_{\mu} a}{a} + \sum_{\mu} \frac{(1 - \cos p_{\mu} a)}{a}$$

= Large mass term
Except for $p_{\mu} = 0$

indicates that the doublers cannot excite (recovering ellipticity) due to heavy mass but chiral symmetry (Z_2 grading) is lost: $\gamma_5 D_W + D_W \gamma_5 \neq 0$.

Nielsen-Ninomiya theorem [1981]

Nielsen-Ninomiya theorem [1981]:

If $\gamma_5 D + D\gamma_5 = 0$, we cannot avoid fermion doubling.

(we have to give up Z_2 grading to recover ellipticity)

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

can avoid NN theorem.

But no concrete form was found in ~20 years.

Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a} (1 + \gamma_5 \text{sgn}(H_W)) \quad H_W = \gamma_5 (D_W - M) \quad M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov} \gamma_5 = a D_{ov} \gamma_5 D_{ov}$

$$\gamma_5 (1 - a D_{ov}/2) \gamma_5 D_{ov} + \gamma_5 D_{ov} \gamma_5 (1 - a D_{ov}/2) = 0.$$

➔ $\Gamma_5 H + H \Gamma_5 = 0.$ = a modified exact chiral symmetry but $\Gamma_5^2 \neq 1.$

$$H = \gamma_5 D_{ov}, \quad \Gamma_5 = \gamma_5 \left(1 - \frac{a D_{ov}}{2} \right)$$

[Luescher 1998]

We can define the index !

[Hasenfratz et al. 1998]

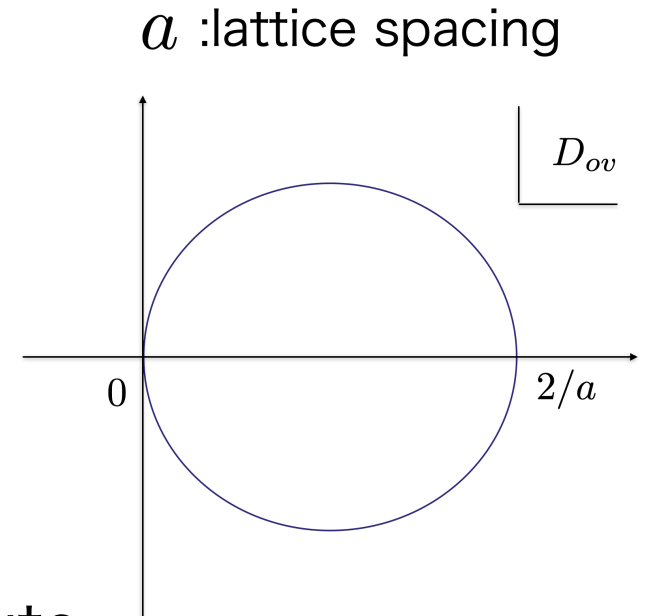
Overlap Dirac spectrum lies on a circle with radius $1/a$

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \gamma_5 \left(1 - \frac{aD_{ov}}{2} \right) \psi_\lambda = 0.$$

(therefore, no contribution to the trace).

The real $2/a$ (doubler poles) do not contribute.



$$\text{Tr} \gamma_5 \left(1 - \frac{aD_{ov}}{2} \right) = \text{Tr}_{\text{zero-modes}} \gamma_5 = n_+ - n_-$$

But D_{ov} is defined with the Wilson Dirac operator.

$$D_{ov} = \frac{1}{a} (1 + \gamma_5 \text{sgn}(H_W)) \quad H_W = \gamma_5 (D_W - M) \quad M = 1/a$$

$$\begin{aligned} \text{Ind} D_{ov} &= \text{Tr} \gamma_5 \left(1 - \frac{a D_{ov}}{2} \right) = \underbrace{\text{Tr} \frac{\gamma_5}{2}}_{=0} - \frac{1}{2} \text{Tr} \text{sgn}(H_W) \\ &= -\frac{1}{2} \text{Tr} \text{sgn}(H_W) \end{aligned}$$

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What is this ???

η invariant of the massive Wilson Dirac operator

$$-\frac{1}{2} \text{Tr} \text{sgn}(H_W) = -\frac{1}{2} \sum_{\lambda_{H_W}} \text{sgn}(\lambda_{H_W}) = -\frac{1}{2} \eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \quad M = 1/a$$

This quantity is known as **the Atiyah-Patodi-Singer η invariant** (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

The Wilson Dirac operator and K-theory

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

$$M = 1/a$$

In this talk, we try to show a deeper mathematical meaning of the right-hand side of the equality,

and try to convince you that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978...]

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What is fiber bundle (for physicists)?

A united manifold of spacetime (= base manifold) and field (fiber)

$$\phi(x) \rightarrow (x, \phi) \in X \times F$$

Spacetime Field space
= base space = fiber space

The direct product structure is realized only locally.
In general, it is “twisted” by gauge fields (connections).

In mathematics, the (isomorphism class of) total space is denoted by E or $E \rightarrow X$

What is fiber bundle? Analogy for (phys) students

X base space (space-time)

= your head

F fiber (field)

= your hair

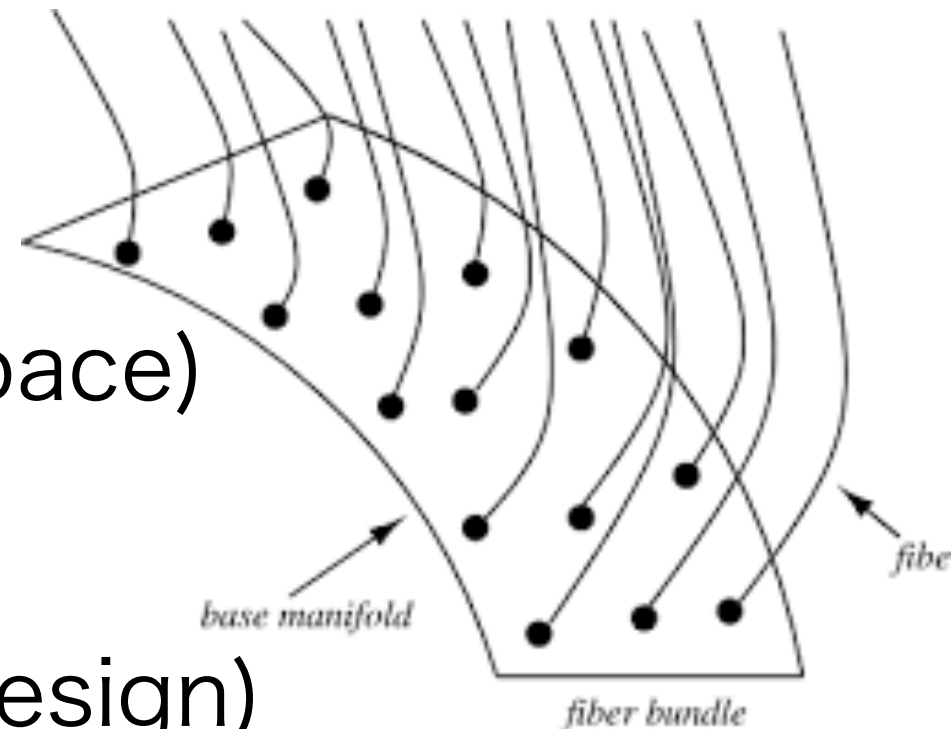
E (= locally $X \times F$) (total space)

= your hair style

Connection

= hair wax (local hair design)

Figure from Wolfram Math world



Classification of **vector bundles**

Let us consider the case $F =$ some vector space.

Compare two vector bundles E_1 and E_2 .

It was proved that the **homotopy theory** can completely classify the vector bundles. **But concrete computation is very difficult.**

K-theory can classify the vector bundles **when their rank is large enough**, detecting some topological invariants.
(more powerful than **the standard** (de Rham) **cohomology theory**).

$K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$
[] denotes the equivalence class (concrete definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \rightarrow E_2 \quad D_{12}^\dagger : E_2 \rightarrow E_1$$

* To be precise, D acts on the sections of E .

to represent the same element by $[E, D, \gamma]$

where

$$E = E_1 \oplus E_2, \quad D = \begin{pmatrix} & D_{12} \\ D_{12}^\dagger & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

* K^0 group describes classification of Dirac operator which anticommutes with chirality operator.

K-theory pushforward

When we are interested in global structure only,
We can forget about details of the base manifold X by taking
“one-point compactification” or the K-theory pushforward :

$$G : K^0(X) \rightarrow K^0(\text{point}) \quad \text{The map just forgets all}$$
$$[E, D, \gamma] \rightarrow [H_E, D, \gamma] \quad \text{but the chiral symmetry.}$$

H_E : The whole Hilbert space on which D acts.

Many information is lost but one (the Dirac operator index) remains.

Suspension isomorphism

The “point” can be suspended to an interval:

There is an isomorphism between 

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

$$[H_E, D, \gamma] \leftrightarrow [p^* H_E, D_t] \quad p^* : \text{pull-back of } p : I \rightarrow \text{point.}$$

we omit in the following.

where the superscript “1” reflects removal of the chirality operator. Instead, the Dirac operator must become one-to-one (no zero mode) at the two endpoints : ∂I

Physical meaning of the isomorphism will be given soon later .

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Atiyah-Singer index

$$\overbrace{n_+ - n_-}^{\text{Ind}(D)} = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu} F_{\rho\sigma})$$

#sol with + chirality #sol with - chirality

Index theorem

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[H_E, D, \gamma] \in K^0(\text{point})$

But we will show that it is isomorphic to

$$[H_E, \gamma(D + m)] \in K^1(I, \partial I)$$

Eigenvalues of continuum massive Dirac operator

$$H(m) = \gamma_5(D_{\text{cont.}} + m) \quad \text{on Euclidean even-dimensional manifold.}$$

Gauge group is U(1) or SU(N)

$$\text{For } D_{\text{cont.}}\phi = 0, \quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$$

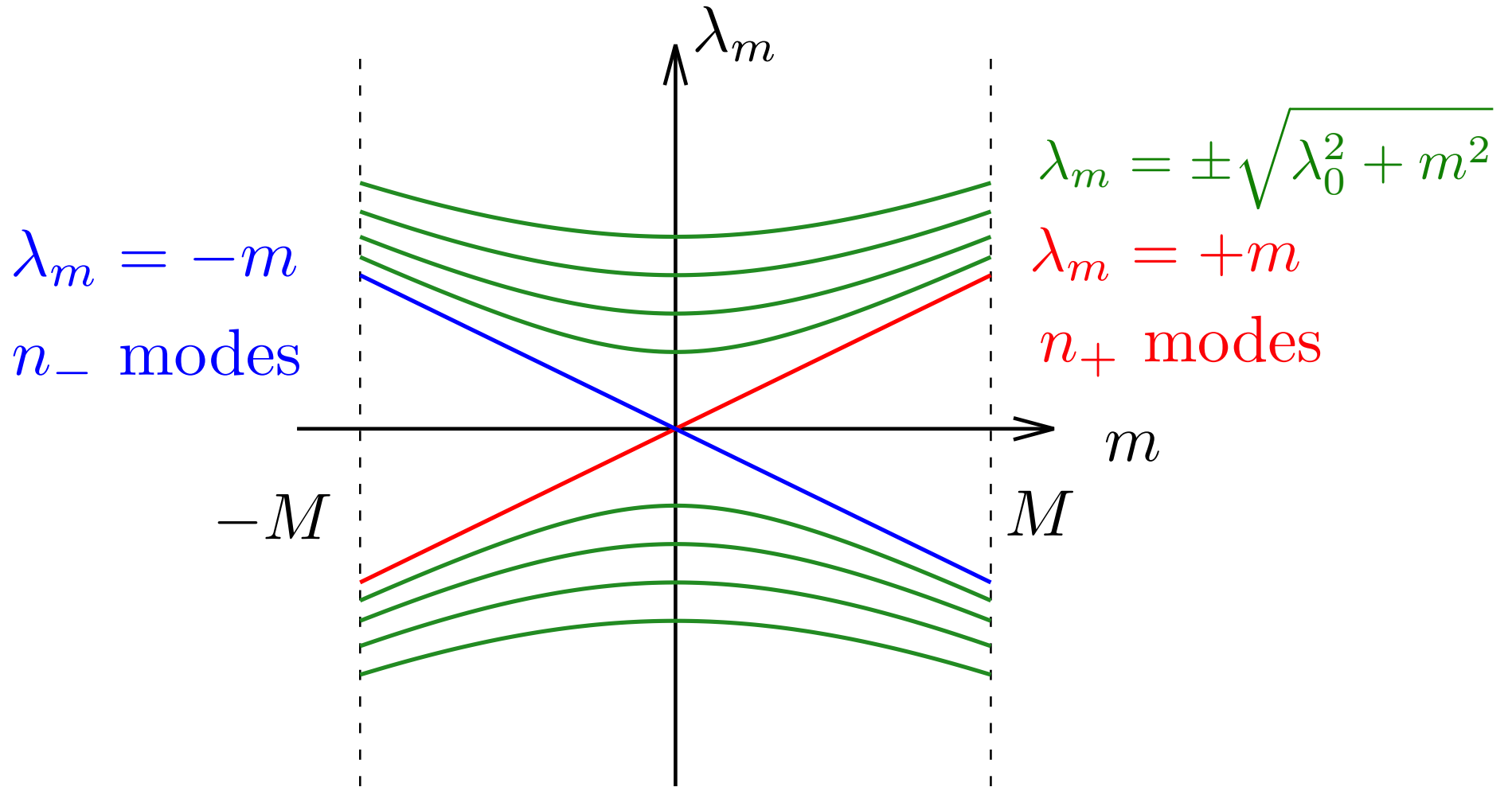
$$\text{For } D_{\text{cont.}}\phi \neq 0, \quad \{H(m), D_{\text{cont.}}\} = 0.$$

$$\text{The eigenvalues are paired: } H(m)\phi_{\lambda_m} = \lambda_m\phi_{\lambda_m}$$

$$H(m)D_{\text{cont.}}\phi_{\lambda_m} = -\lambda_m D_{\text{cont.}}\phi_{\lambda_m}$$

$$\text{As } H(m)^2 = -D_{\text{cont.}}^2 + m^2, \quad \text{we can write them } \lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$$

Spectrum of $H(m) = \gamma_5(D_{\text{cont.}} + m)$



Spectral flow = Atiyah-Singer index = η invariant

n_+ = # of zero-crossing eigenvalues from - to + $H(m) = \gamma_5(D_{\text{cont.}} + m)$

n_- = # of zero-crossing eigenvalues from + to -

$n_+ - n_- =:$ **spectral flow** of $H(m)$ $m \in [-M, M]$

Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

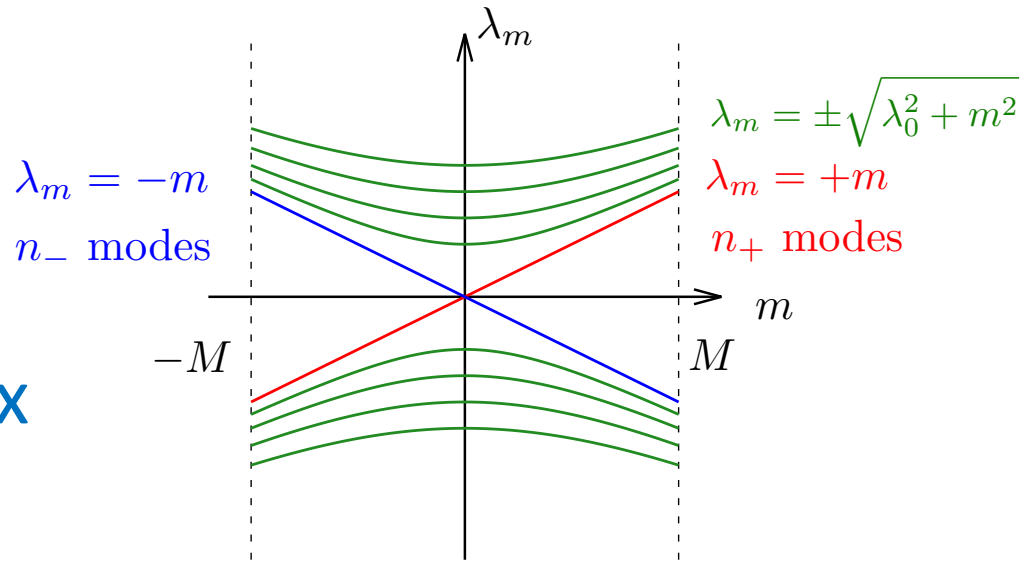
$$\eta(H) = \sum_{\lambda \geq 0}^{\text{reg}} - \sum_{\lambda < 0}^{\text{reg}}$$

$$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

Suspension isomorphism in K theory

Massless=
counting index
by points



Massive=
counting
index by lines

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

point

line=interval

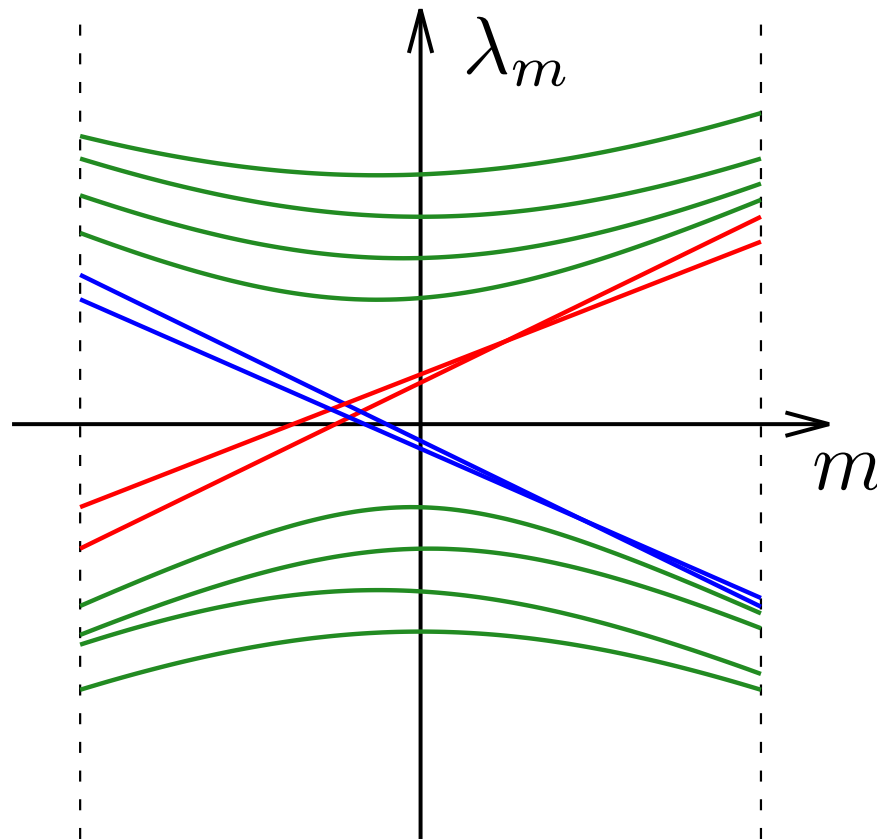
with chirality operator

without chirality operator

⇒ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (**massless**) is difficult but counting lines (**massive**) still works.

Standard definition:
Where is $m=0$?
What are zero modes?



Eta invariant:
If $m = \pm M$ points are gapped, we can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

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Counting lines (massive, K^1) is easier than counting points (massless, K^0).

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Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: **2n-dimensional flat torus** T^{2n}

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with **gauge field** A_i

D : Dirac operator on sections of E

$$D_{\text{cont.}} = \gamma_i (\partial_i + A_i)$$

Chirality (Z_2 grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

Wilson Dirac operator on a lattice

We regularize T^{2n} is by a **square lattice with lattice spacing** a

(The fiber is still continuous.)

We denote the bundle by E^a and

link variables :

$$U_k(\mathbf{x}) = P \exp \left[i \int_0^a A_k(\mathbf{x}') dl \right],$$

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right]$$

Wilson term

$$a \nabla_i^f \psi(\mathbf{x}) = U_i(\mathbf{x}) \psi(\mathbf{x} + \mathbf{e}_i) - \psi(\mathbf{x})$$

$$a \nabla_i^b \psi(\mathbf{x}) = \psi(\mathbf{x}) - U_i^\dagger(\mathbf{x} - \mathbf{e}_i) \psi(\mathbf{x} - \mathbf{e}_i)$$

Note: In our paper, we consider "generalized link variables" to determine the gauge fields both in continuum and on a lattice simultaneously. But the standard Wilson line works, too.

Definition of $K^1(I, \partial I)$ group

Let us consider a Hilbert bundle with

Base space $I = \text{range of mass } [-M, M]$

boundary $\partial I = \pm M$ points

Fiber space $\mathcal{H} = \text{Hilbert space to which } D \text{ acts}$

D_m : one-parameter family labeled by m .

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by equivalence classes of the pairs:

$[(\mathcal{H}, D_m)]$ **having the same spectral flow.**

Note: K^1 group does **NOT** require any chirality operator.

Definition of $K^1(I, \partial I)$ group

Group operation: $[(\mathcal{H}^1, D_m^1)] \pm [(\mathcal{H}^2, D_m^2)] = [(\mathcal{H}^1 \oplus \mathcal{H}^2, \begin{pmatrix} D_m^1 & \\ & \pm D_m^2 \end{pmatrix})]$

Identity element: $[(\mathcal{H}, D_m)]|_{\text{Spec.flow}=0}$

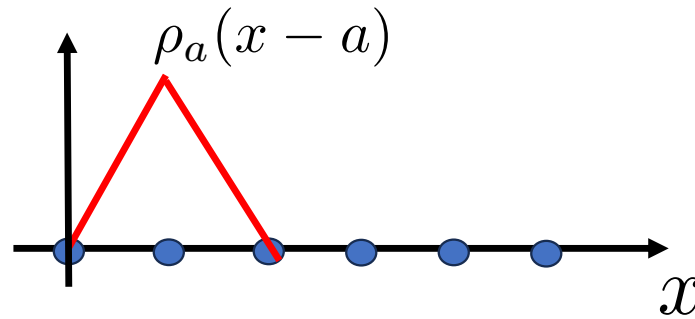
We compare $[(\mathcal{H}_{\text{cont.}}, \gamma(D_{\text{cont.}} + m))]$ and $[(\mathcal{H}_{\text{lat.}}, \gamma(D_W + m))]$

taking their difference, and confirm if **the lattice-continuum combined Dirac operator**

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & f_a \\ f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

has Spectral flow =0 where f_a^* f_a are “**mixing mass term**” with some “nice” mathematical properties.

$$f_a : H^{\text{lat.}} \rightarrow H^{\text{cont.}}$$



maps from **finite-dimensional** Hilbert space on a discrete lattice to **infinite-dimensional** continuum one :

$$f_a \phi(x) := a^n \sum_{z \in \text{lattice sites}} \rho_a(x - z) U(x, z) \phi(z).$$

$U(x, z)$: parallel transport (or Wilson line) to ensure the gauge invariance.

$\rho_a(x - z)$: weight function (multi-) linearly interpolating the nearest-neighbors.

To control the norm before/after the map, it satisfies

$$\int_{x \in T^n} \rho_a(x - z) d^n x = 1 \qquad a^n \sum_{z \in \text{lattice sites}} \rho_a(x - z) = 1.$$

$$f_a^* : H^{\text{cont.}} \rightarrow H^{\text{lat.}}$$

Is defined by

$$f_a^* \psi_1(z) := \int_{x \in T^n} \rho_a(z - x) U(x, z)^{-1} \psi_1(x) d^n x.$$

Note) $f_a^* f_a$ is not the identity but smeared around nearest-neighbor sites.
(The gauge invariance is maintained by the Wilson lines.)

Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i ,
a constant c exists such that

$$\|D_i \phi\|^2 \leq c(\|\phi\|^2 + \|D\phi\|^2)$$

When a covariant derivative is large, D is also large.

This property is nontrivial on a lattice.

$$\|\nabla_i^f \phi\|^2 \leq c(\|\phi\|^2 + \|D_W \phi\|^2)$$

Without Wilson term, doubler modes would have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important to make the Dirac operator elliptic.

Continuum limit of $f_a^* f_a$

1. For arbitrary $\phi^{\text{lat.}}$

$$\lim_{a \rightarrow 0} f_a \phi^{\text{lat.}} \text{ weakly converges to a } \exists \phi_0^{\text{cont.}} \in L_1^2$$

where L_1^2 is a subspace of $H^{\text{cont.}}$ where the elements and their **first derivatives are square integrable**.

$$2. \lim_{a \rightarrow 0} f_a \gamma(D_W + m) \phi^{\text{lat.}} \text{ weakly converges to } \gamma(D + m) \phi_0^{\text{cont.}} \in L^2$$

$$3. \text{ There exists } c \text{ s.t. } \|f_a^* f_a \phi^{\text{lat.}} - \phi^{\text{lat.}}\|_{L^2}^2 < ca^2 \|\phi^{\text{lat.}}\|_{L_1^2}^2$$

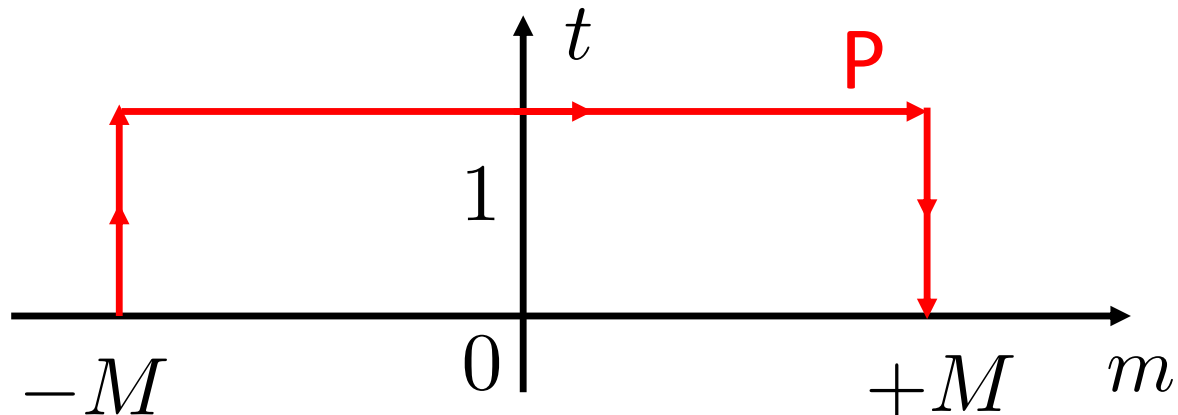
$$4. \text{ For any } \phi^{\text{cont.}} \in L_1^2, \quad \lim_{a \rightarrow 0} f_a f_a^* \phi_0^{\text{cont.}} = \phi_0^{\text{cont.}}$$

Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$$

on the path P :



Main theorem

There exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & tf_a \\ tf_a^* & -\gamma(D_W + m) \end{pmatrix}$$

is invertible (having no zero mode) on the staple-shaped path P

[which is a sufficient condition for Spec.flow=0]

$\Rightarrow \gamma(D_{\text{cont.}} + m), \gamma(D_W + m)$ have the same spec.flow

$$\Rightarrow \frac{1}{2}\eta(\gamma(D - M))^{\text{PV reg.}} = \frac{1}{2}\eta(\gamma(D_W - M))$$

The continuum and lattice indices agree.

Proof (by contradiction)

Assume $\hat{D} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m) & t f_a \\ t f_a^* & -\gamma(D_W + m) \end{pmatrix}$

has zero mode(s) at arbitrarily small lattice spacing.

\Rightarrow For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j} + m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

Continuum limit

Multiplying $\begin{pmatrix} 1 \\ f_{a_j} \end{pmatrix}$ and taking the continuum limit

$$\begin{pmatrix} \gamma(D_{\text{cont.}} + m_\infty) & t_\infty \\ t_\infty & -\gamma(D_{\text{cont.}} + m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

u_∞, v_∞ are L_1^2 weakly convergent

$$\hat{D}_\infty^2 = D_{\text{cont.}}^2 + m_\infty^2 + t_\infty^2$$

requires

$$m_\infty = t_\infty = 0.$$

L^2 strongly convergent
(Rellich's theorem)

Contradiction with $m^2 + t^2 > 0$ along the path P.

Numerical test

We consider a two-dimensional square lattice (or torus)

We set link variables as

$$U_y(x, y) = \exp \left[i \frac{2\pi Q(x - x_0)a}{L_1^2} \right]$$

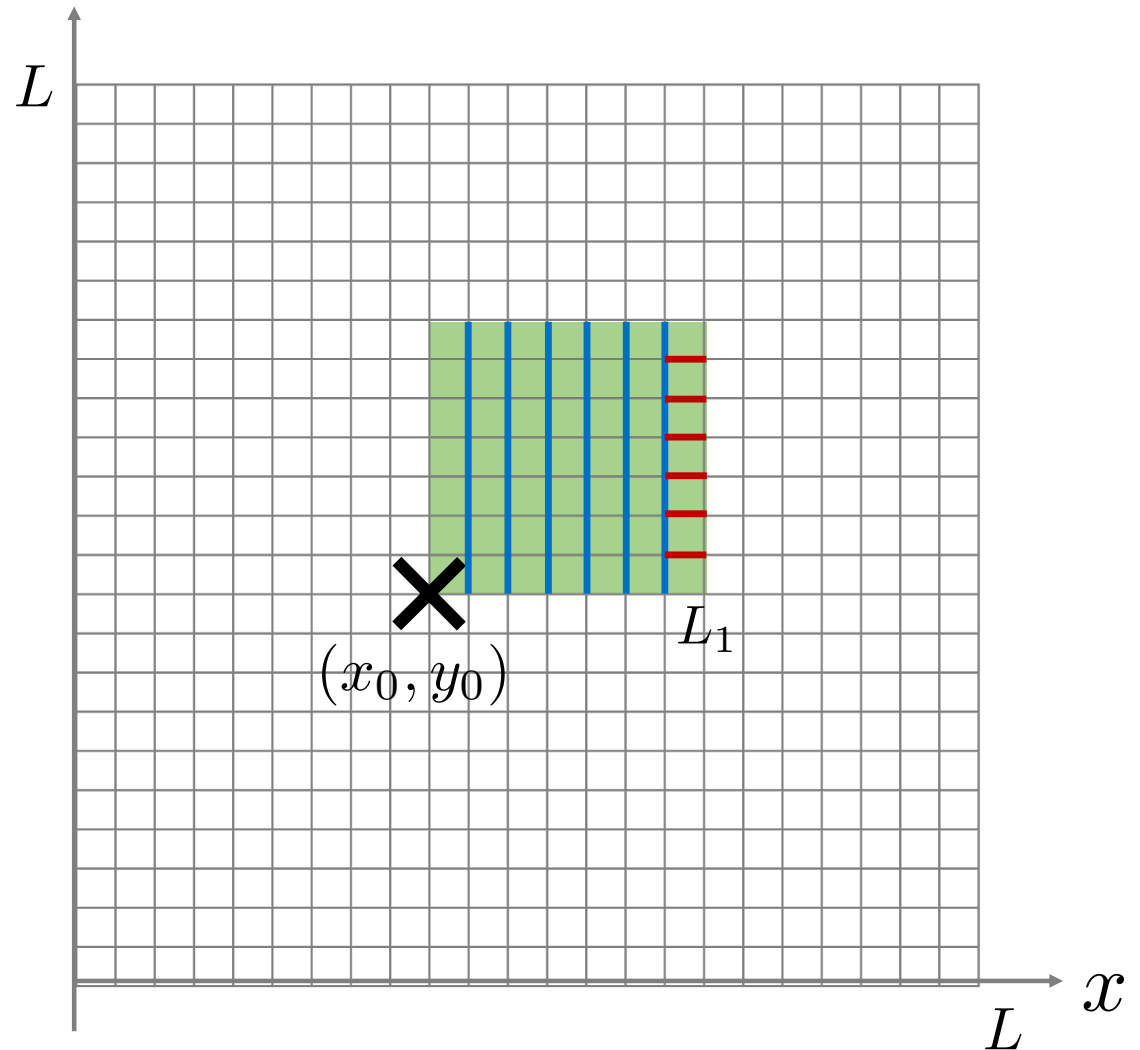
$$U_x(x, y) = \exp \left[-i \frac{2\pi Q(y - y_0)}{L_1} \right]$$

others = 1.

Then every green plaquette has a constant curvature

$$U_P(x, y) = \exp \left[i \frac{2\pi Q a^2}{L_1^2} \right]$$

so that **geometrical index will be Q**.



This constant curvature background can be extended to any even dimensions with SU(N) gauge connections [Cf. Hamanaka-Kajiura 2002].

Massive Wilson Dirac

$$\gamma D_W(m) = \gamma \left[\sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right] + m \right]$$

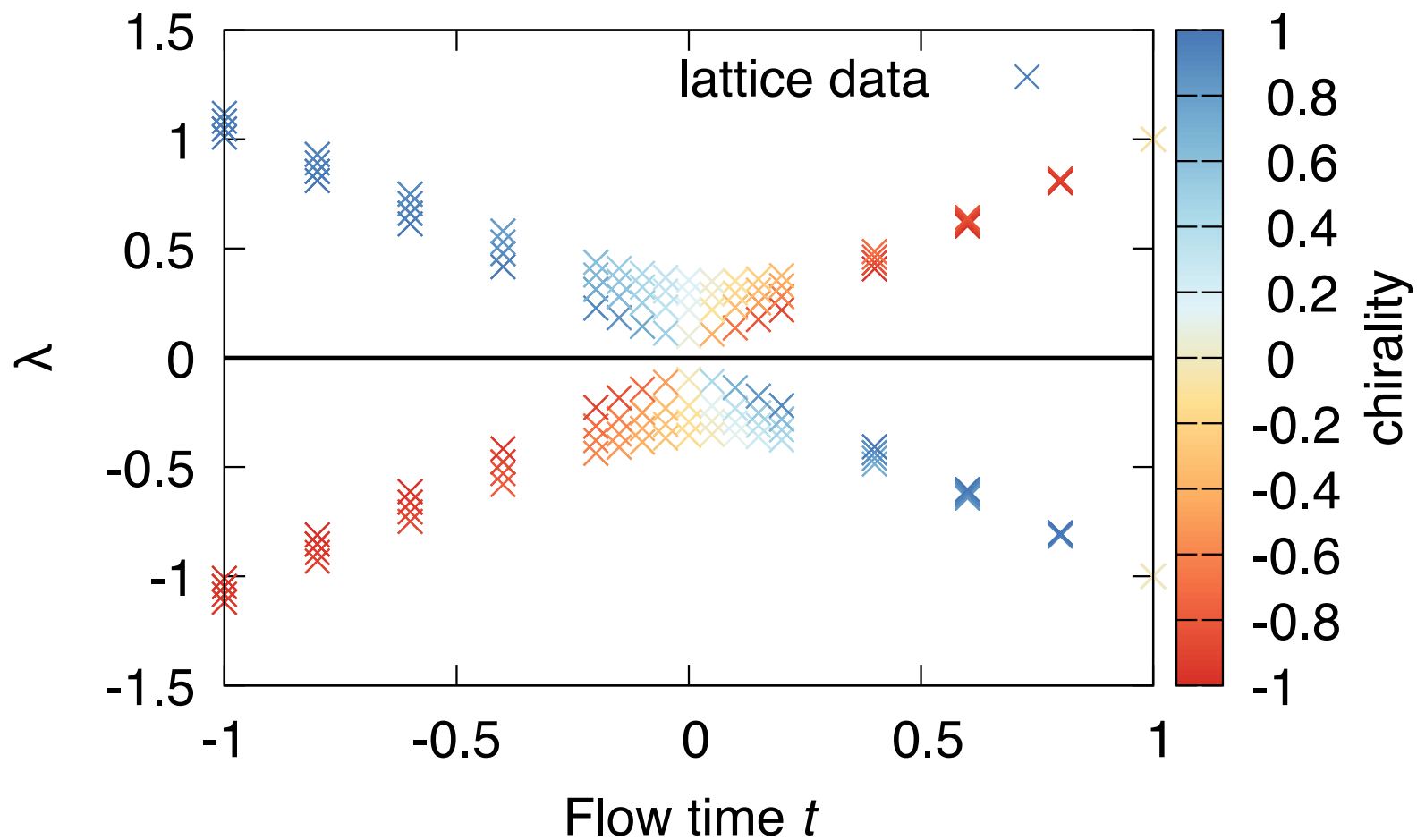
$$a \nabla_i^f \psi(\mathbf{x}) = U_i(\mathbf{x}) \psi(\mathbf{x} + \mathbf{e}_i) - \psi(\mathbf{x}) \quad a \nabla_i^b \psi(\mathbf{x}) = \psi(\mathbf{x}) - U_i^\dagger(\mathbf{x} - \mathbf{e}_i) \psi(\mathbf{x} - \mathbf{e}_i)$$

with periodic b.c. in x-direction and anti-periodic b.c. in y direction. We set $L=32$ and $L1=10$.

We compute near-zero eigen-spectrum

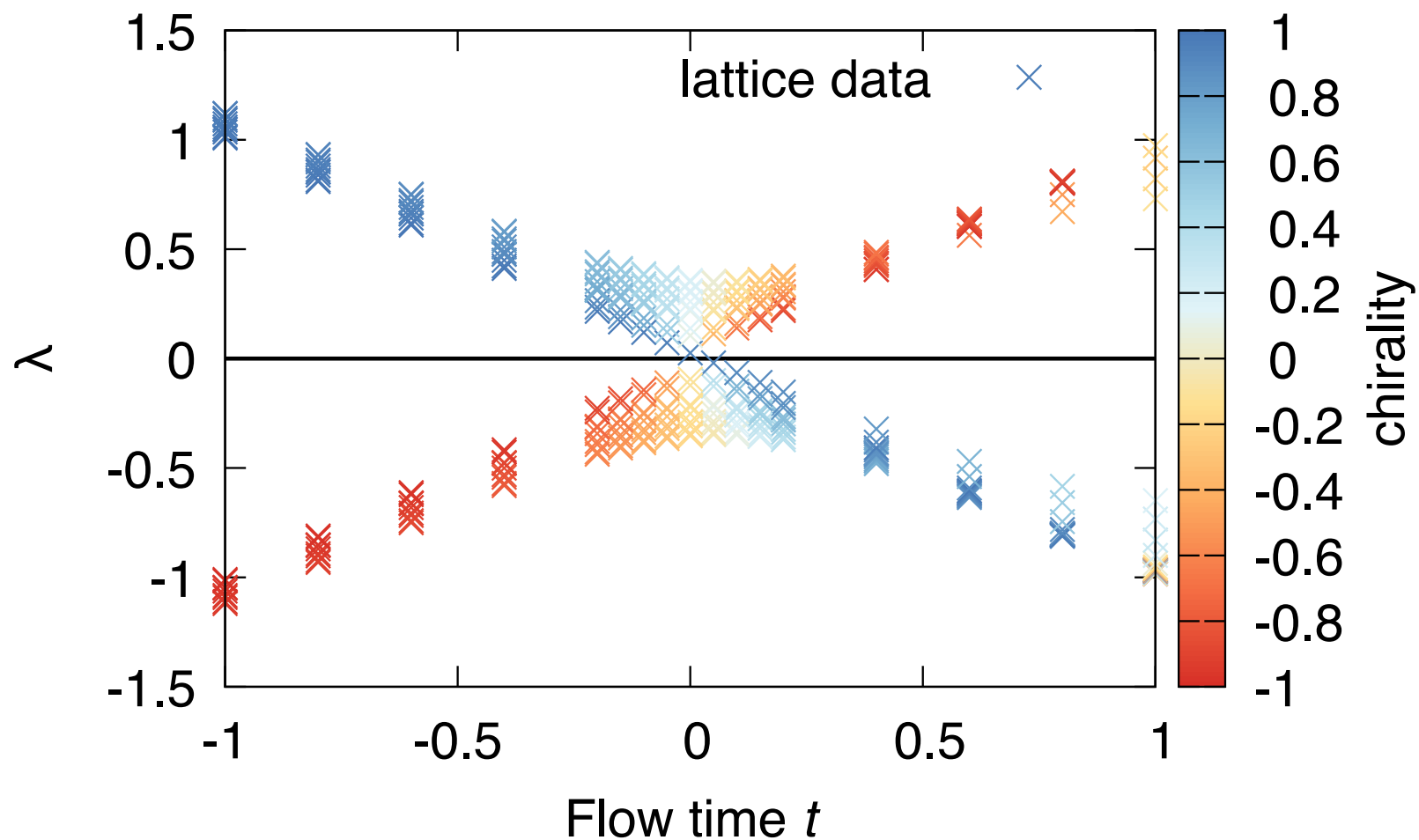
in the range $-1 \leq m \leq +1$

Numerical result at $Q=0$



There is no
zero crossing :
index=0.

Numerical result at $Q=+1$



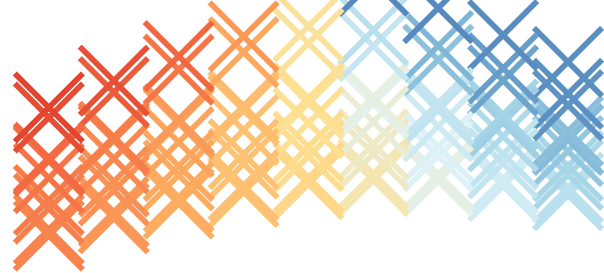
There is one crossing from positive to negative:
index=+1.

Numerical result at $Q=+1$

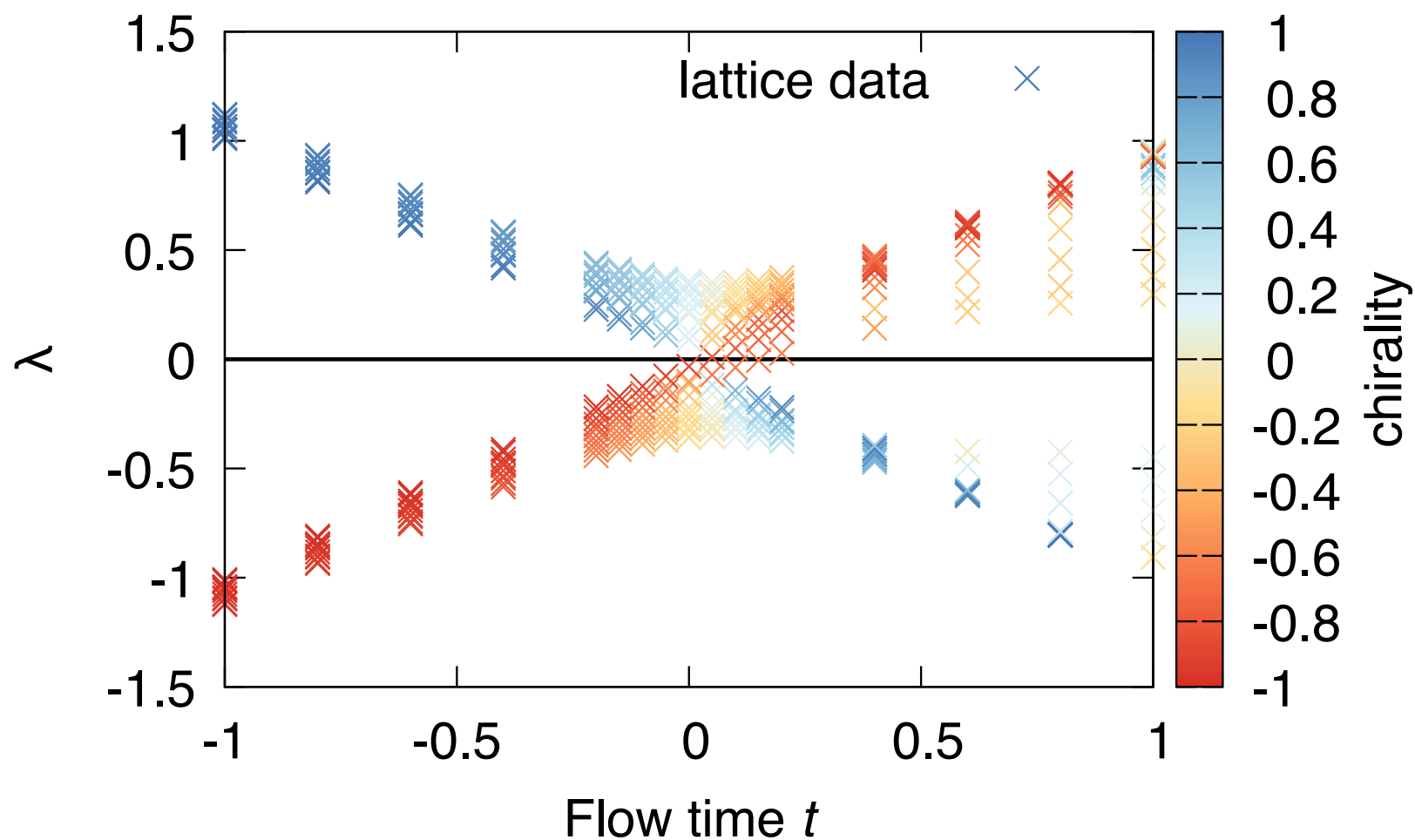


There is one crossing from positive to negative:

index=+1.

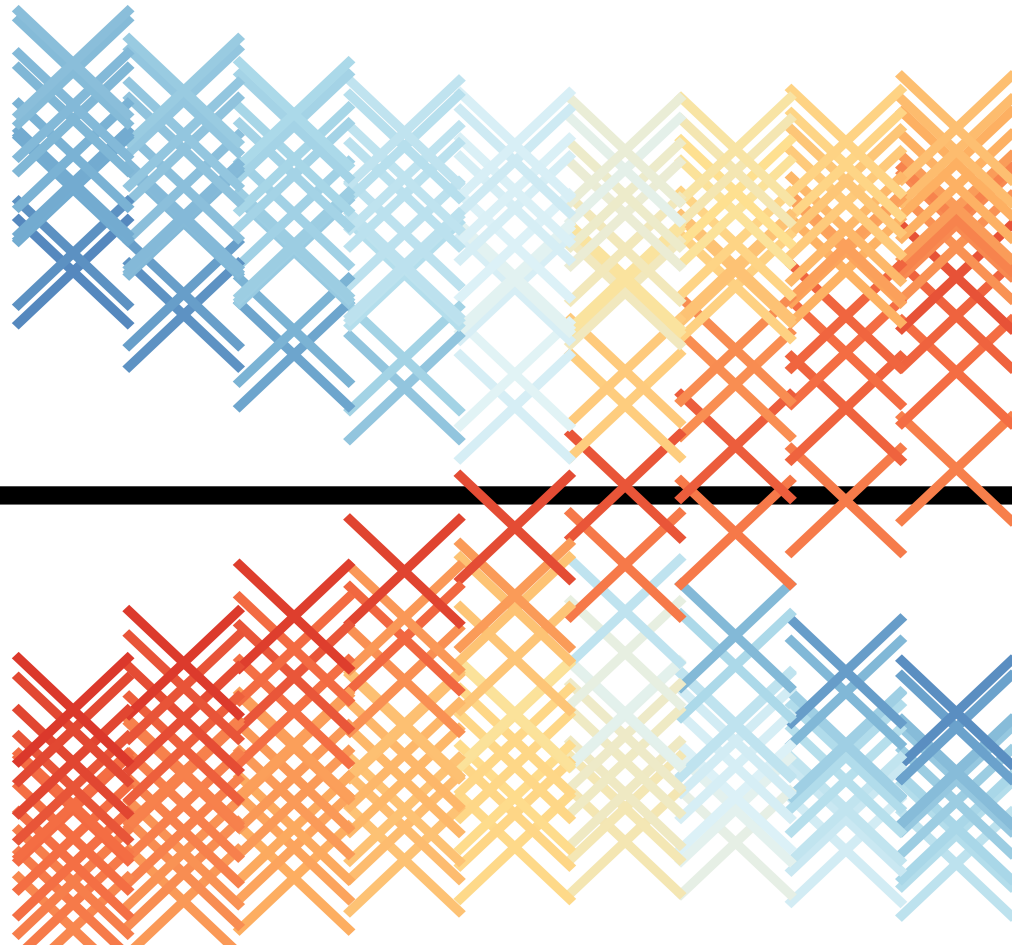


Numerical result at $Q=-2$



There is two crossings from negative to positive:
index=-2.

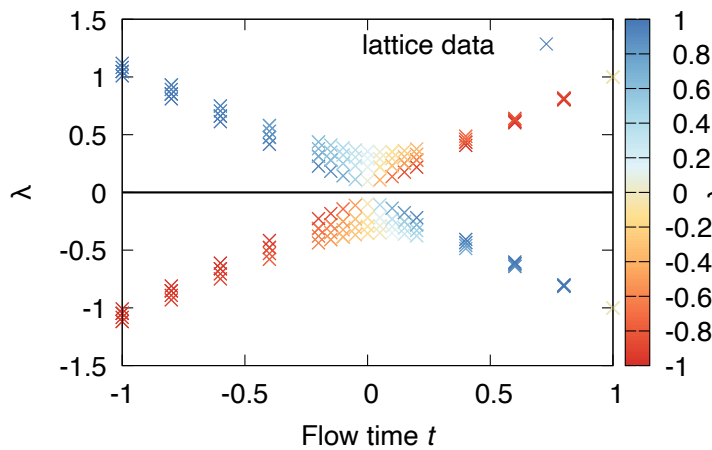
Numerical result at $Q=-2$



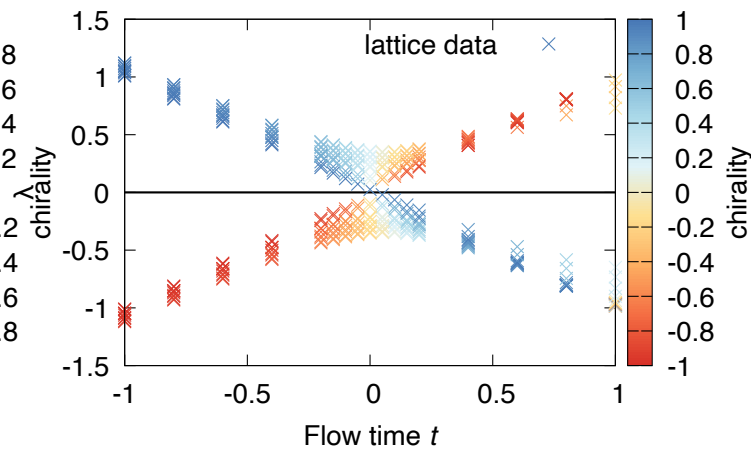
There are two crossings from negative to positive:
index=-2.



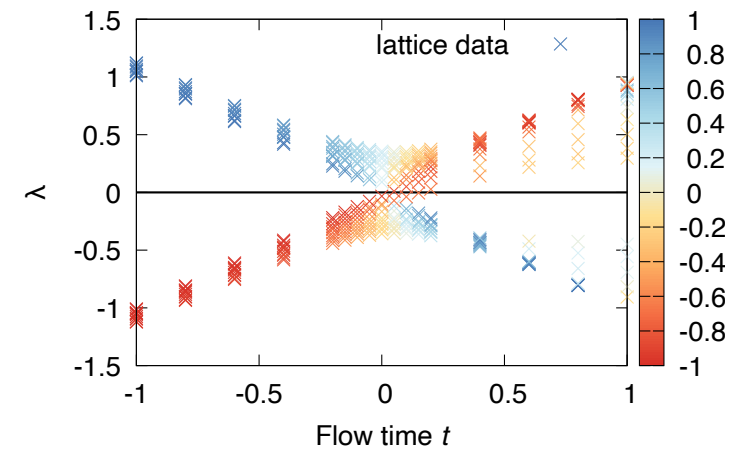
Our 32x32 lattice reproduces the Atiyah-Singer index theorem on a torus.



Index= $Q=0$



Index= $Q=+1$



Index= $Q=-2$

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✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)

is great but equivalent to the eta invariant of the massive Wilson Dirac op.

✓ 3. K-theory

classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

✓ 4. Massless Dirac (K^0 group) vs. massive Dirac (K^1 group) in continuum

Counting lines (massive, K^1) is easier than counting points (massless, K^0).

✓ 5. Main theorem on a lattice

The proof is given by lattice-continuum combined Dirac operator, which is gapped.

6. Comparison with the overlap Dirac index

7. Summary and discussion

Wilson Dirac operator is **equally good** as D_{ov} to describe the index.

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$

By $K^1(I, \partial I)$
for sufficiently small
lattice spacings

Suspension
isomorphism

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Suspension
isomorphism

Or even better?

Application to the manifolds with boundaries

Periodic b.c.

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$

Dirichlet b.c. (Shamir domain-wall fermion) we can show

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = -\frac{1}{2}\eta(\gamma_5(D_{DW}^{\text{cont.}})) = \text{Ind}_{\text{APS}}D^{\text{cont.}}$$

[perturbative equality F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019].

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita 2019].

But the **overlap Dirac is missing** because Ginsparg-Wilson relation is broken by the boundary [Luescher 2006].

Real Dirac operators and the mod-two index

For general complex Dirac operators,

$$K^1(I, \partial I) \rightarrow -\frac{1}{2}\eta(H_W) = -\frac{1}{2}\eta(\gamma_5(D - M))$$

For real Dirac operators, for example, in SU(2) gauge theory in 5D (origin of Witten anomaly), we will be able to show

$$\begin{aligned} KO^1(I, \partial I) &\rightarrow -\frac{1}{2} \left[1 - \text{sgn det} \left(\frac{D_W - M}{D_W + M} \right) \right] = -\frac{1}{2} \left[1 - \text{sgn det} \left(\frac{D_{\text{cont.}} - M}{D_{\text{cont.}} + M} \right) \right] \\ &= \text{Ind}_{\text{mod-two}} D_{\text{cont.}} \end{aligned}$$

But there is no overlap Dirac counterpart.

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we expect wider applications (to APS boundary and real case) than the overlap index.

7. Summary and discussion

Summary

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978...]

Summary

$$\text{Ind}D_{ov} = -\frac{1}{2}\eta(H_W) = \overset{K^1(I, \partial I)}{\downarrow} -\frac{1}{2}\eta(\gamma_5(D_{\text{cont.}} - M)) = \text{Ind}D_{\text{cont.}}$$
$$H_W = \gamma_5(D_W - M)$$

We have shown a deeper mathematical meaning of the right-hand side of the equality,

and that the massive Wilson Dirac operator is an equally good or even better object than D_{ov} to describe the gauge field topology in terms of K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978...]

Backup slides

What are the weak convergence and strong convergence?

The sequence v_j weakly converges to v_∞ when for arbitrary w

$$\lim_{j \rightarrow \infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j \rightarrow \infty} (v_j - v_\infty)(x) \rightarrow \lim_{k \rightarrow \infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j \rightarrow \infty} \|v_j - v_\infty\|^2 = 0$.

Rellich's theorem:

$$L_1^2 \text{ weak convergence} = L^2 \text{ convergence}$$