Domain-wall fermion and index theorem

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work in progress with M. Furuta (U. Tokyo), S. Matsuo (Nagoya U.), T. Onogi(Osaka U.), S. Yamaguchi (Osaka U.), M. Yamashita (U.Tokyo)

Cf. HF,. T Onogi, S. Yamaguchi PRD96(2017) no.12, 125004 [arXiv:1710.03379]

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What's new in this talk

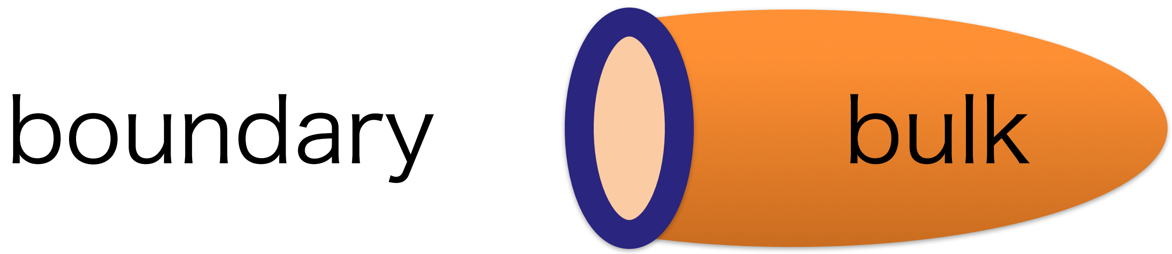
In 2017 Dec 8, I gave a seminar here, about
"A physicist-friendly reformulation of the Atiyah-Patodi-Singer index theorem."

F,. Onogi, Yamaguchi PRD96(2017) no.12, 125004 [arXiv:1710.03379]

Today, I will talk about its mathematical proof.

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, in progress

Atiyah-Patodi-Singer (APS) index theorem [1975]



$$Ind(D_{\mathrm{APS}}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

curvature

* Here we (mainly) consider 4-dimensional flat Euclidean space with boundary at $x_4=0$.

APS index in topological insulator

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

Witten 2015 : APS index is a key to understand bulk-edge correspondence in (Time-reversal (T)) symmetry protected topological insulator:

fermion path integrals

$$Z_{\mathrm{edge}} \propto \exp(-i\pi\eta(iD^{\mathrm{3D}})/2)$$ T-anomalous

$$Z_{\mathrm{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$ T-anomalous

$$Z_{\mathrm{edge}}Z_{\mathrm{bulk}} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$ ➡ T is protected !

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16, Freed-Hopkins 16, Witten 16, Yonekura 16…]

What puzzled us

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   [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

What puzzled us

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→ We launched a study group reading original APS paper and it took 3 months to translate it into "physics language" and we found another fermionic quantity, which coincides with the APS index [FOY 2017].

A physicist-friendly reformulation using domain-wall fermion

[F, Onogi, Yamaguchi 2017]

$$\mathfrak{I} = \frac{1}{2}\eta(H_{DW})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

$$H_{DW} = \gamma_5(D_{\rm 4D} + M\epsilon(x_4))$$

⬇ perturbative computation

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

coincides with APS index, keeping the features of topological insulator.

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.

Mathematicians joined us.

In August, I gave a talk to Mikio Furuta (U. Tokyo).

He said “Interesting!”

Moreover, only 1 week later,

he proposed a proof of

$$\frac{1}{2}\eta(H_{DW}^{reg}) = Ind(D_{\mathrm{APS}})$$

[F, Furuta, Matsuo, Onogi,

Yamaguchi, and Yamashita,

in progress.]

Overview

$$\mathfrak{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

|| || in physicist's sense

$Ind(D_{\mathrm{APS}})$
with physicist-unfriendly
boundary condition [APS 1975]

=

$\frac{1}{2}\eta(H_{DW})$
with physicist-friendly
set-up (topological insulator)
[FOY 2017]

THIS WORK!

[FMOYY 2019]

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2. Physicist's view of index theorems
3. Atiyah-Singer index with massive fermion operator
4. Index with domain-wall fermion Dirac operator [F, Onogi, Yamaguchi 2017]
5. Mathematical proof [this work]
6. Discussion
7. Summary

Atiyah-Singer index theorem [1968] on a manifold without boundary

A theorem on the number of solutions of Dirac equation $D\psi = 0$

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\overbrace{(F_{\mu\nu}F_{\rho\sigma})}^{\boldsymbol{E}\cdot\boldsymbol{B}}$$

n_+: #sol with + chirality

n_-: #sol with - chirality

we consider U(1) or SU(N) gauge field (connection).

Dirac equation = EOM of electrons

Schrodinger equation (non-relativisitic)

$$\left[i\frac{\partial}{\partial t}+\frac{1}{2m}\frac{\partial^2}{\partial x_i^2}\right]\psi=0.$$

Klein-Gordon equation (consistent only for bosons) $\left[-\partial_t^2+\partial_i^2+m^2\right]\psi=0.$

Dirac equation $\left[-i\gamma_\mu\partial^\mu+m\right]\left[i\gamma_\mu\partial^\mu+m\right]\psi=0.$

$$\left[i\gamma_\mu\partial^\mu+m\right]\psi=0.$$

Dirac operator: $D=\gamma^\mu(\partial_\mu+iA_\mu)$

Gamma matrices and chirality

$$D = \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

gamma matrices space-time derivatives EM field (connection)

4x4 gamma matrices in Euclidean 4D space

$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$$

σ_i Pauli matrices

Chirality operator

$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Algebra

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} \qquad \{\gamma_5, \gamma_{\nu}\} = 0$$

$$\mathrm{Tr}\gamma_5\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma} = 4i\epsilon_{\mu\nu\rho\sigma} \qquad \mathrm{Tr}\gamma_5(\text{up to 3 } \gamma\text{'s}) = 0$$

Chirality = spin in moving direction

Left-handed fermion has $\gamma_5 = -1$

Right-handed has $\gamma_5 = 1$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1^{os} \\ |\uparrow\rangle \\ |\downarrow\rangle \\ \psi_4^{os} \end{pmatrix}$$

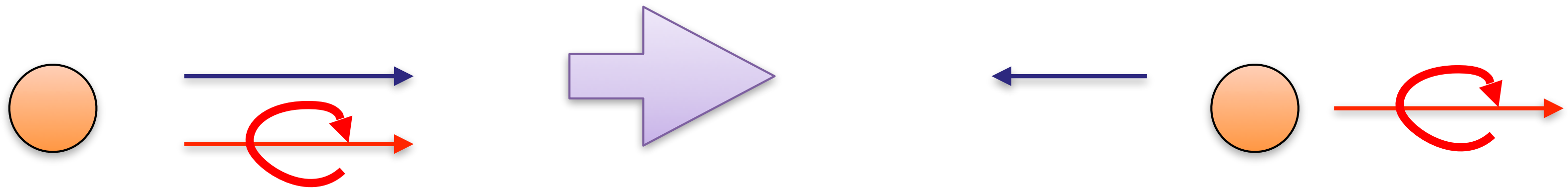
* os=off-shell modes, non-classical, not satisfying

$$E = mc^2\sqrt{1+\boldsymbol{p}^2/m^2c^2}$$

but this is true only for massless fermion.

Chirality = spin in moving direction

For massive fermion, we can flip the chirality by Lorentz transformation,



but for massless fermion (with speed of light) we cannot.

Naively, for the index theorem, fermion needs to be massless.

Atiyah-Singer index

$$D = \gamma^\mu(\partial_\mu + iA_\mu) \qquad \{D, \gamma_5\} = 0.$$

$$\gamma_5\phi(x) = +\phi(x) \to \gamma_5 D\phi(x) = -D\gamma_5\phi(x) = -D\phi(x)$$

Eigenmodes make ± pairs

except for zero-modes:

$$n_+ - n_- = \mathrm{Tr}\gamma_5^{\mathrm{reg}}.$$

#sol with + chirality #sol with - chirality

Physicist-friendly description (Fujikawa method 1979)

1. Heat-kernel regularization

$$\mathrm{Tr}\gamma_5^{\mathrm{reg}} = \lim_{M\to\infty} \mathrm{Tr}\gamma_5 e^{\frac{D^2}{M^2}}$$

2. plane-wave complete set

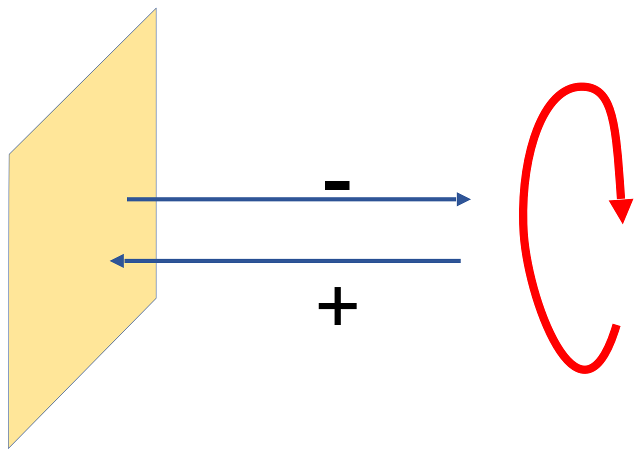
$$= \lim_{M\to\infty} \int d^4x \int d^4k\, e^{-ikx} \mathrm{tr}\gamma_5 e^{D_{4\mathrm{D}}^2/M^2} e^{ikx}$$

3. perturbative expansion $\left(D^2 = D_\mu D^\mu + \frac{g}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)$

$$= \frac{g^2}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$[\gamma_5, A] = 0.$$

$$\Rightarrow \quad \text{index} = n_+ - n_-$$

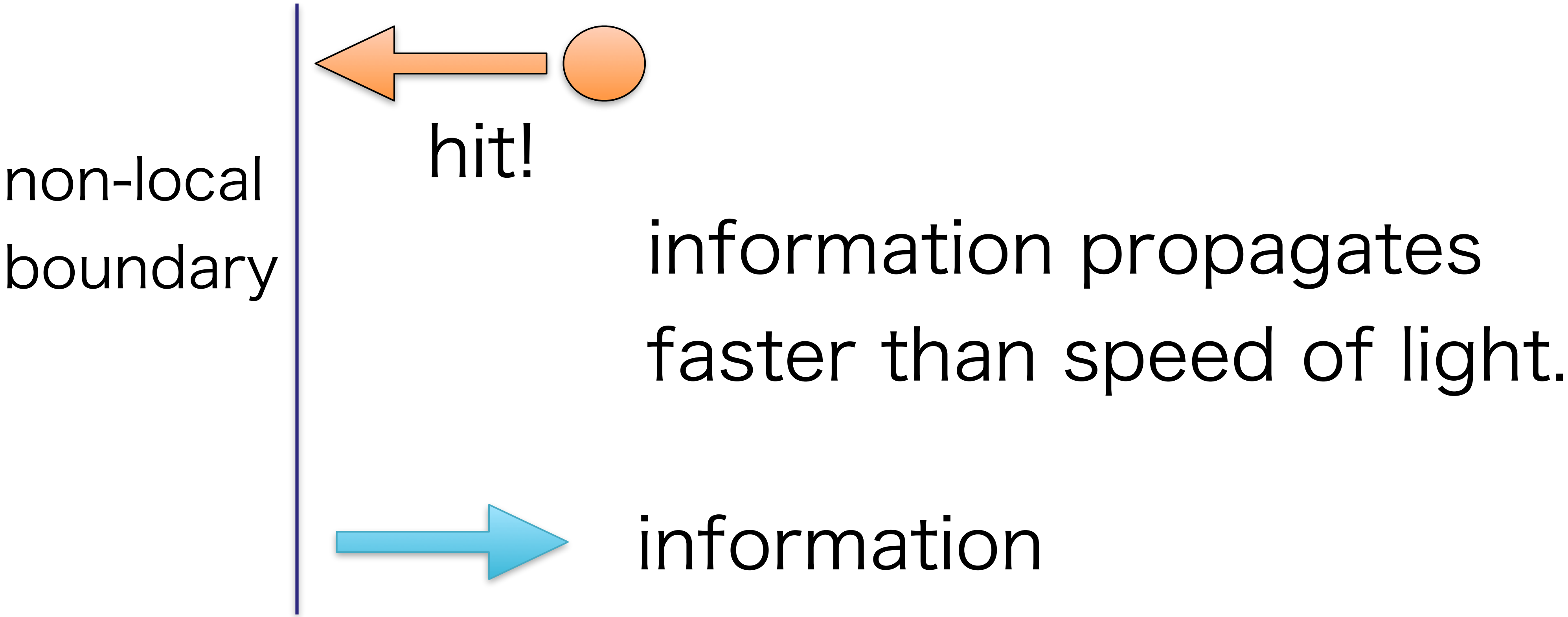
Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.



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→ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is “Yes, we can”.

Contents

Massive Dirac operator

$$D + M = \begin{pmatrix} M & D_{LR} \\ D_{RL} & M \end{pmatrix}$$

Anti-Hermitian Hermitian

(propotional to identity matrix)

Let's consider a Hermitian operator:

$$H = \gamma_5(D + M) \qquad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4.$$

on a manifold without boundary.

Zero-modes & non-zero modes

$$H = \gamma_5(D + M) \qquad D = \gamma^\mu(\partial_\mu + iA_\mu)$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M\phi_0 = \pm M\phi_0.$$

Non-zero modes make ± pairs

$$H\phi_i = \lambda_i\phi_i$$

$$HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

Eta invariant of massive Dirac operator

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

$$H = \gamma_5(D + M)$$

$$= \#\ \mathrm{of}\ +M - \#\ \mathrm{of}\ -M$$

coincides with the original AS index!

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coincides with the original AS index!

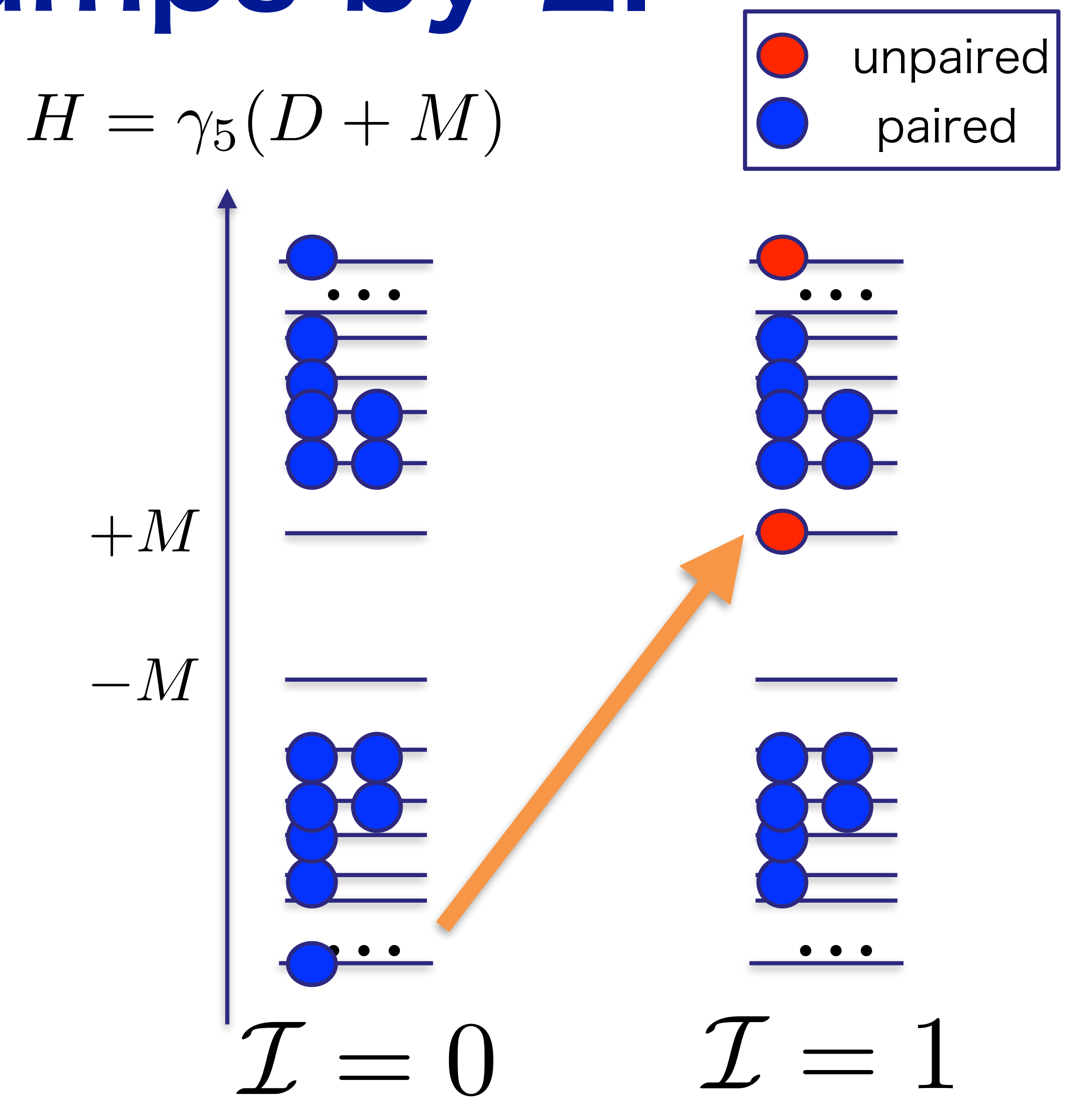
In fact, we need a factor 1/2.

$$\mathrm{Index}(D) = \frac{1}{2}\eta(H)^{reg}.$$

$\eta(H)$ always jumps by 2.

To increase + modes,
we have to borrow
one from - (UV) modes.

Good regularizations
(e.g. Pauli-Villars, lattice)
respect this fact.



$$\Rightarrow \quad \text{Index}(D) = \frac{1}{2}\eta(H).$$

Perturbative "proof" (in physics sense)

using Pauli-Villars regulator

$$\frac{1}{2}\eta(H)^{reg} = \frac{1}{2}\left[\eta(H) - \eta(H_{PV})\right].$$

$$H = \gamma_5(D+M)$$

$$H_{PV} = \gamma_5(D+\Lambda), \quad \Lambda \gg M$$

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\frac{H}{(\sqrt{H^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr}He^{-tH^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(\mathrm{sgn}M + \frac{D}{M}\right)e^{-t'D^\dagger D/M^2}e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

$$= \mathrm{sgn}M\frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

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4. Index with domain-wall fermion Dirac operator [F, Onogi, Yamaguchi 2017]

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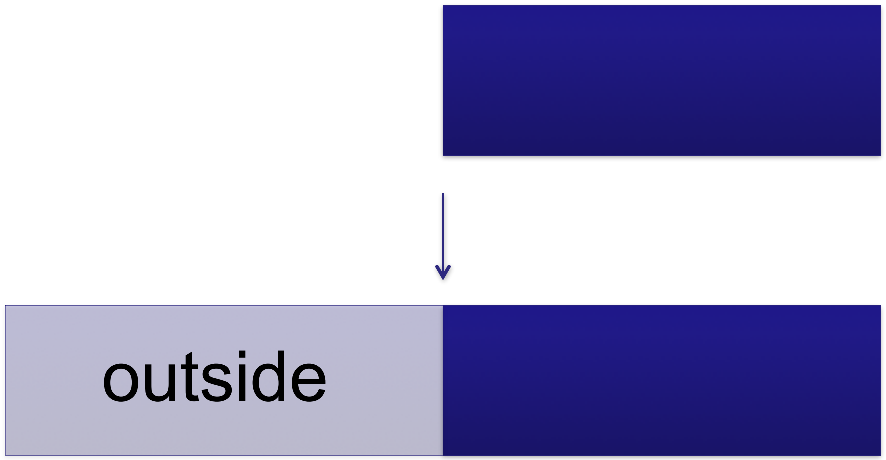
More physical set-up?

In physics,

1. Any boundary has "outside": manifold + boundary → domain-wall.



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2. Boundary should not preserve helicity but keep angular-mom: massless $\to$ massive (in bulk)

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3. Boundary condition should not be put by hand → but automatically chosen.

More physical set-up?

In physics,

1. Any boundary has "outside": manifold + boundary → domain-wall.
2. Boundary should not preserve helicity but keep angular-mom: massless → massive (in bulk)
3. Boundary condition should not be put by hand → but automatically chosen.
4. Edge-localized modes play the key role.

Domain-wall Dirac operator

Let us consider

$$D_{\rm 4D} + M\epsilon(x_4), \quad \epsilon(x_4) = {\rm sgn} x_4$$

[Jackiw-Rebbi 1976, Callan-Harvey 1985, Kaplan 1992]

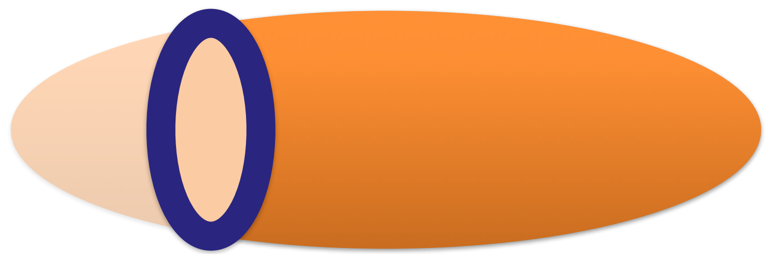
on a closed manifold

with sign flipping mass,

without assuming any

boundary condition

(we expect it dynamically given.).

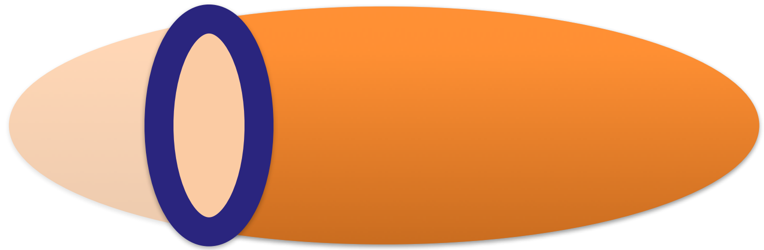


“new” APS index

[F-Onogi-Yamaguchi 2017]

$$\mathfrak{I} = \frac{\eta(H_{DW}^{reg})}{2}$$

$$\left(= \frac{\eta(H_{DW})}{2} - \frac{\eta(H_{PV})}{2}\right)$$



$$H_{DW} = \gamma_5(D_{4\mathrm{D}} + \underline{M\epsilon(x_4)})$$

$$\epsilon(x_4) = \mathrm{sgn} x_4$$

$$H_{PV} = \gamma_5(D_{4\mathrm{D}} - M_2)$$

Fujikawa method

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

coincides with APS index !

PV part = Atiyah-Singer index

$$\eta(H_{PV}) = \lim_{s\to 0} \mathrm{Tr} \frac{H_{PV}}{(\sqrt{H_{PV}^2})^{1+s}} = \frac{1}{\sqrt{\pi}} \int_0^\infty dt t^{-1/2} \mathrm{Tr} H_{PV} e^{-tH_{PV}^2}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty dt' t'^{-1/2} \mathrm{Tr} \gamma_5 \left(-1 + \frac{D}{M} \right) e^{-t' D^\dagger D / M^2} e^{-t'},$$

Fujikawa-method

does not contribute.

$(t' = M^2 t)$

$$= -\frac{1}{32\pi^2} \int d^4x \, \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}_c F^{\mu\nu} F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

$$H_{PV} = \gamma_5 (D_{4\mathrm{D}} - M_2)$$

Domain-wall fermion part

Now let's compute

$$\eta(H_{DW}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{DW}}{(\sqrt{H_{PV}^2})^{1+s}} = \lim_{s\to 0}\frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt t^{(s-1)/2}\mathrm{Tr}H_{DW}e^{-tH_{DW}^2}$$

$$H_{DW} = \gamma_5(D_{4\mathrm{D}} + M\epsilon(x_4))$$

In the free fermion case,

$$H_{DW}^2 = -\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4).$$

→ eigenvalue problem = Schrodinger equation with δ-function-like potential.

Complete set in the free case

Solutions to $\left(-\partial_{x_4}^2 + \omega^2 - 2M\gamma_4\delta(x_4)\right)\varphi = 0$

are

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\text{edge}}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$

→ Edge mode appears !

where $\omega = \sqrt{\boldsymbol{p}^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,\text{edge}}_{\pm,e/o} = \pm\varphi^{\omega,\text{edge}}_{\pm,e/o}$

3D direction = conventional plane waves.

“Automatic” boundary condition

We didn’t put any boundary condition by hand. But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\Big|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the δ-function-like potential.

This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Fujikawa-method

$$\eta(H_{DW}) = \frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt' t'^{\frac{s-1}{2}} \mathrm{Tr}\gamma_5 \left(\epsilon(x_4) + \frac{D}{M}\right) e^{-t' H_{DW}^2/M^2} e^{-t'},$$

Perturbative expansion

We insert our complete set $\{\varphi^{\omega,\mathrm{edge}}_{\pm,e/o}(x_4) \times e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\}$

(See our paper for the details.)

100% edge-mode effect

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{3\mathrm{D}})$$

$$\epsilon(x_4) = \mathrm{sgn} x_4 \qquad\qquad = (CS \text{ mod integer})$$

Total index

$$\mathfrak{I} = \frac{\eta(H_{DW}))}{2} - \frac{\eta(H_{PV})}{2}$$

$$= \frac{1}{2}\left[\frac{1}{32\pi^2}\int d^4x\ \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{\mathrm{3D}})\right.$$

$$\left. + \frac{1}{32\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}\right]$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})$$

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✔ 4. Index with domain-wall fermion Dirac operator [F, Onogi, Yamaguchi 2017]
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5. Mathematical proof [this work]

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This talk =

A mathematical proof for

$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW}^{reg})$$

on general even-dimensional manifold.

APS

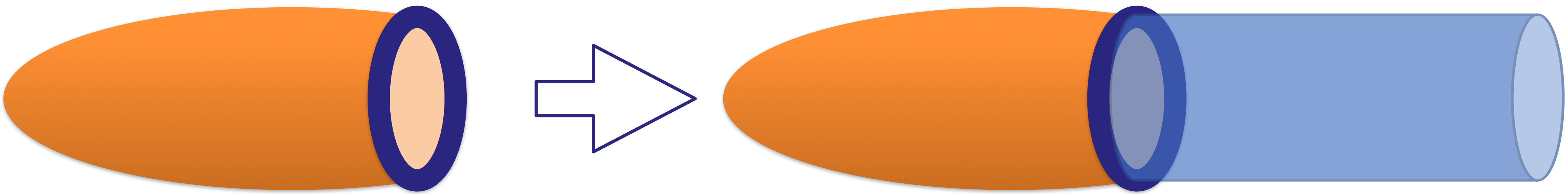
1. massless Dirac
   (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(2) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary

Domain-wall fermion

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed



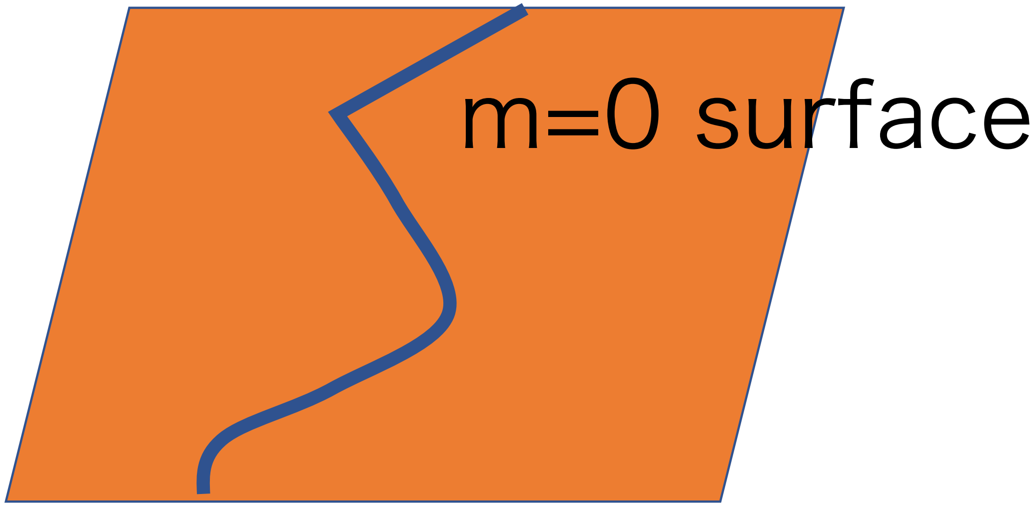
Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent "mass", we can localize the zero modes to "massless" lower-dimensional surface and the index is given by the product:

$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) =$$
$$Ind(D^d) \times Ind(\gamma_s \partial_s + M(s))$$



= generalization of domain-wall fermion

Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D_{\rm APS}^{odd-dim}) = \frac{1}{2}\left[\eta(D^{\rm boundary1}) - \eta(D^{\rm boundary2})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_s + \gamma_5(D^{4\mathrm{D}} + m(s,x)) \\ -\partial_s + \gamma_5(D^{4\mathrm{D}} + m(s,x)) & 0 \end{pmatrix}$$

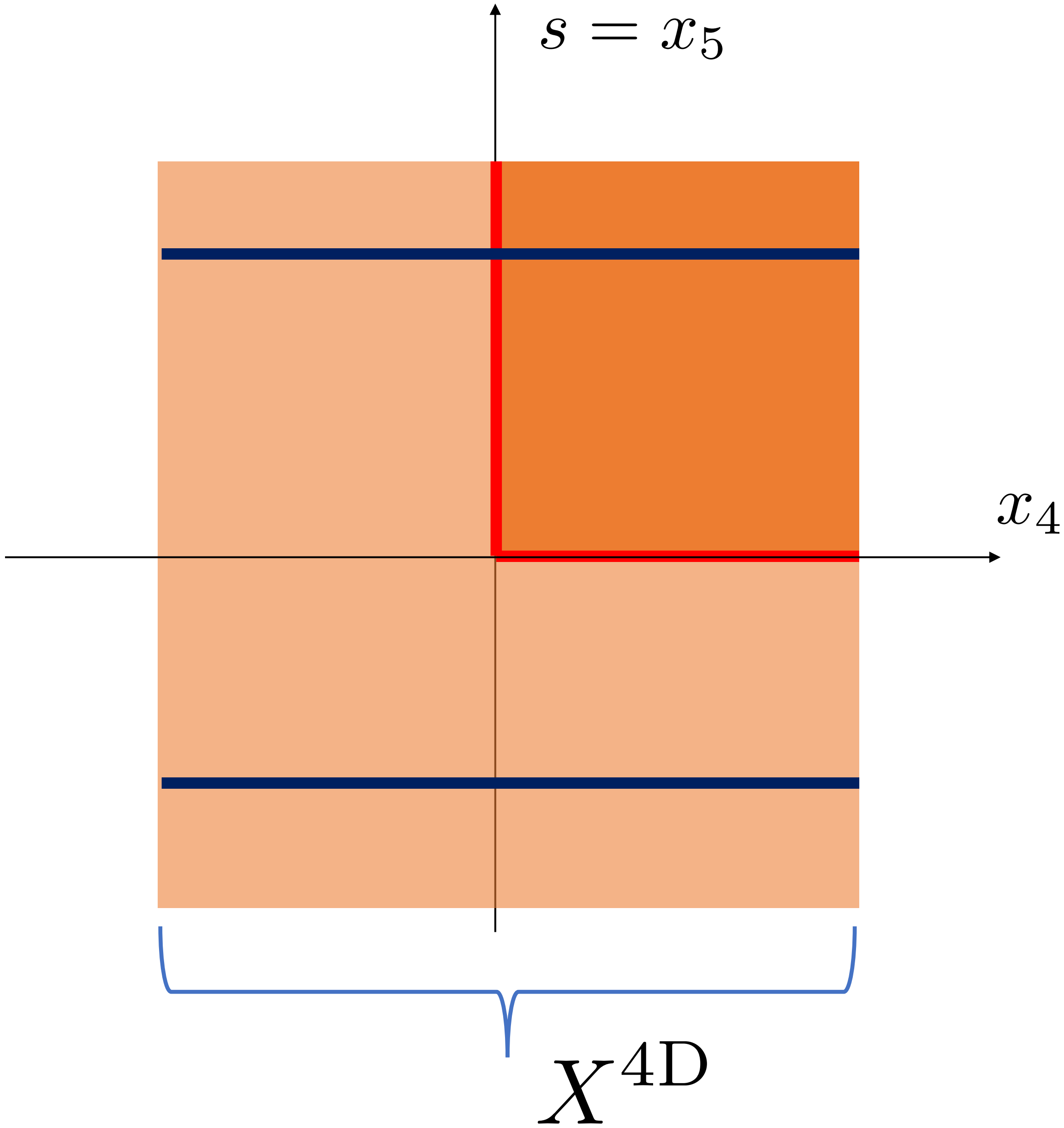
where

$$m(x,s) = \begin{cases} M & \text{for } x_4 > 0 \ \& \ s > 0 \\ 0 & \text{for } x_4 = 0 \ \& \ s = 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of s,

on X^{4D} x R,



and compute

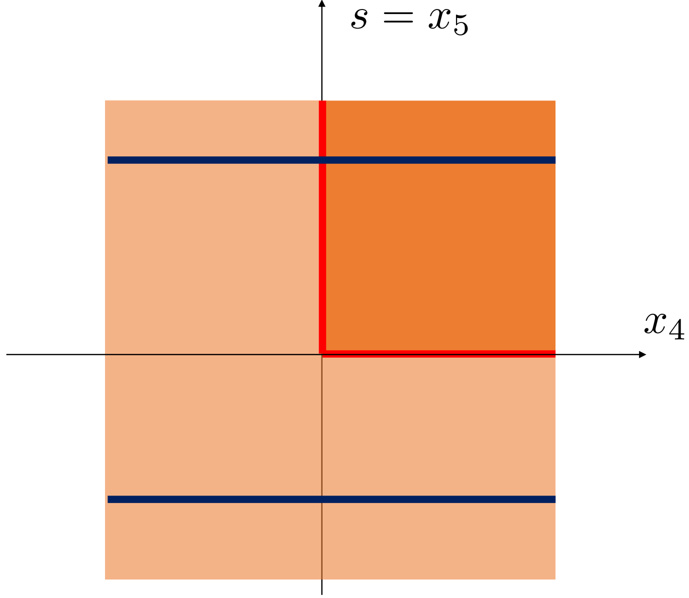
$$Ind(D^{5D})$$

in two different ways:

1. localization

2. eta-inv. at $s = \pm 1$.

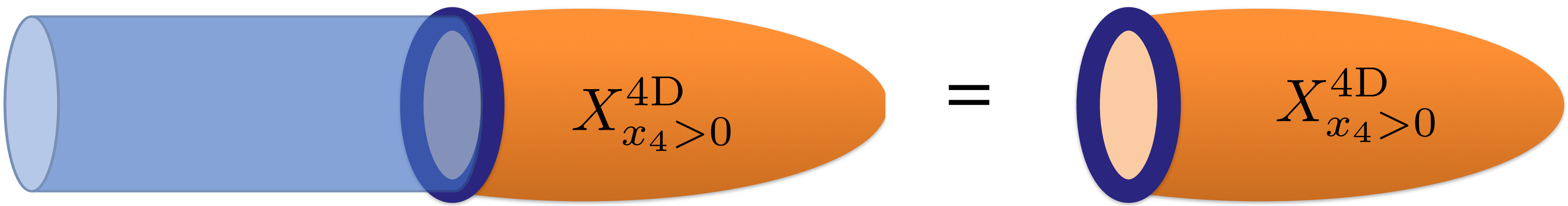
Localization



Theorem 2 tells us

$$Ind(D^{5D})|_{M,M_2\to\infty} = Ind(D^{4D}_{m=0\mathrm{surface}}) \times \underbrace{Ind D^{1D}_{normal}}_{=1}$$

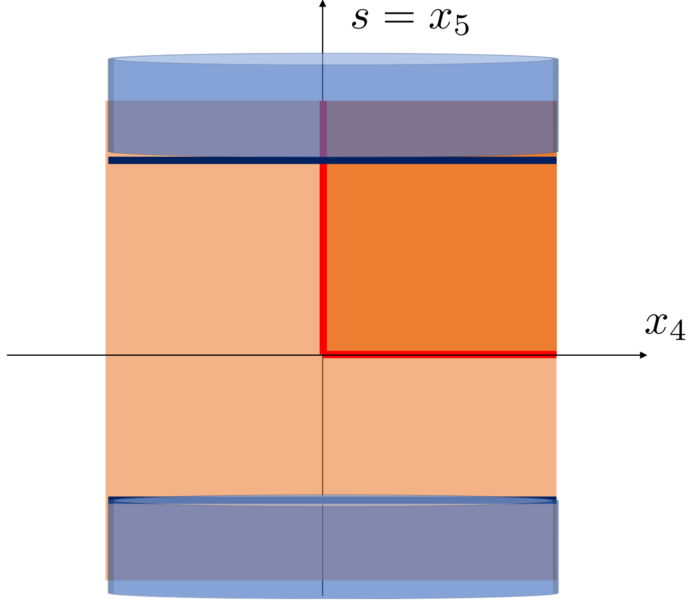
and on the massless surface



theorem 1 indicates

$$Ind(D^{4D}_{m=0\mathrm{surface}}) = Ind(D^{X^{4D}_{x_4>0}}_{\mathrm{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{5\mathrm{D}}) = Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}at s=\pm 1})$$

and from theorem 3, we obtain

$$Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}at s=\pm 1}) = \frac{1}{2}\left[\eta(D^{4\mathrm{D}}_{s=1}) - \eta(D^{4\mathrm{D}}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4)) - \eta(\gamma_5(D^{4\mathrm{D}} - M_2)\right] = \frac{1}{2}\eta^{PV reg.}(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4))$$

therefore,

$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H^{reg}_{DW})$$

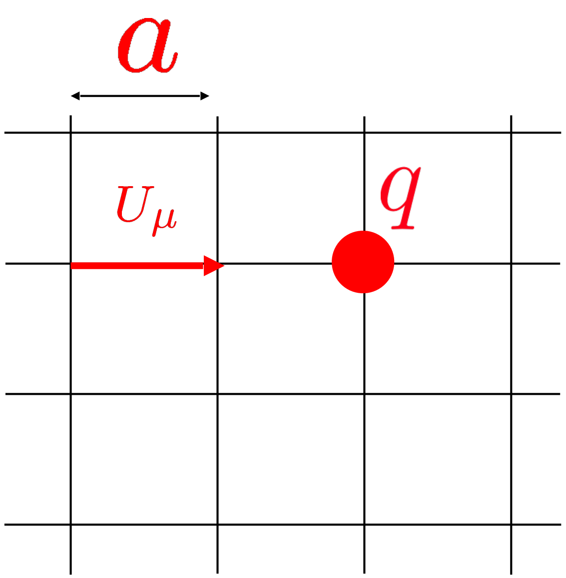
Q.E.D.

Contents

My main subject = lattice gauge theory.

$$U_{n,\mu} = \exp(igaA_\mu(n + \hat{\mu}/2))$$

$$L = \beta \sum_{\mu,\nu=1}^{4} \mathrm{Tr}[U_{n,\mu} U_{n+\mu,\nu} U^\dagger_{n+\nu,\mu} U^\dagger_{n,\nu}] + \bar{q}_n \left[\sum_\mu \gamma_\mu \frac{U_{n,\mu} q_{n+\hat{m}u} - U^\dagger_{n-\hat{m}u,\mu} q_{n-\hat{\mu}}}{2a} + m \right] q_n$$



Oakforest-PACS at U. of Tsukuba

On lattice, Dirac equation is a difference equation.

$$\frac{\partial}{\partial x}\psi(x) \to \frac{\psi(x+a)-\psi(x)}{a}$$

For $Ind(D_{\rm APS})$, we do not know how to realize the APS boundary condition.

But $$\frac{1}{2}\eta(\gamma_5(D^{\rm lat}+M\epsilon(x_4)))$$

is easy and guaranteed to be an integer!

[F, Kawai, Matsuki, Onogi, Yamaguchi, in progress.]

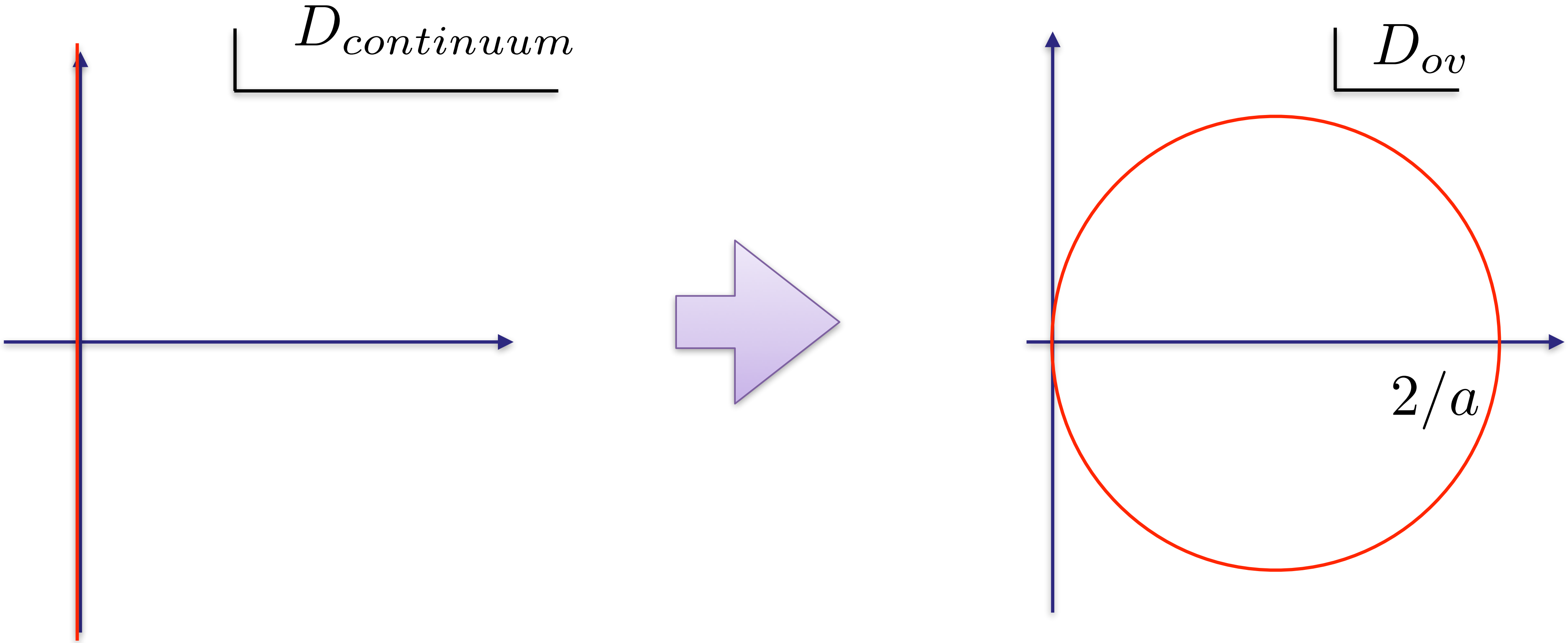
How about 5D bulk and 4D boundary?

It was known that 5D lattice Dirac operator ends up (after removing bulk effect) with a 4D lattice Dirac operator (overlap Dirac operator)

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}\gamma_5 (D^{lat} - 1/a)\right)$$

of which index is Atiyah-Singer index on 4D manifold + O(a^2) errors.

Lattice AS index = index in K-theory by Karoubi?



We need more communication between mathematicians and physicists!

Summary

1. APS index describes bulk-edge correspondence of topological insulators.
2. APS (as well as AS) index can be reformulated by massive domain-wall Dirac operator.
3. We have given a mathematical proof for general cases.
4. math<->phys communication is important.

Backup slides

Example : 1+1d bulk + 0+1d edge Majorana fermion coupled to gravity

APS index tells

$$Z \propto \exp\left(2\pi i \frac{n}{8}\right)$$

consistent with Z_8 classification of Kitaev's interacting Majorana chain.

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{3\mathrm{D}})}{2} = \frac{CS}{2\pi} + \mathrm{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x\ \mathrm{tr}_c\left[\epsilon_{\nu\rho\sigma}\left(A^\nu\partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$