グラディエントフローの基礎とその応用

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Outline

Notations

- The spacetime signature is $(+,+,+,+)$ (euclidean) and gamma matrices are all hermitian.
- We normalize the gauge group generator as

$$\mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}.$$

The structure constant are defined by

$$[T^a, T^b] = f^{abc} T^c,$$

and quadratic Casimirs are

$$f^{acd} f^{bcd} = C_A \delta^{ab}, \qquad \mathrm{tr}(T^a T^b) = -T\delta^{ab}, \qquad T^a T^a = -C_F \mathbb{1}.$$

- We also use the following abbreviation for the momentum integral:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}.$$

Yang–Mills gradient flow (Lüscher (2009–))

- Yang–Mills gradient flow is an evolution of the gauge field $A_\mu(x)$ along a fictitious time $t \in [0, \infty)$, according to

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

  where

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

  and its initial value is the conventional gauge field

$$B_\mu(t=0,x) = A_\mu(x).$$

- RHS is the Yang–Mills equation of motion, the gradient in function space if S_{YM} is regarded as a potential height. So the name of the gradient flow.

Yang–Mills gradient flow (Lüscher (2009–))

- This is a sort of diffusion equation in which the diffusion length is

$$x \sim \sqrt{8t}.$$

- The flow makes the field configuration smooth; it generates the smearing/cooling for a lattice gauge field.
- A theoretical understanding on its renormalizability (see below) however distinguishes the gradient flow from other smearing/cooling methods.

Yang–Mills gradient flow

- Yang–Mills gradient flow (continuum)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\rm YM}}{\delta B_\mu(t,x)}, \qquad B_\mu(t=0,x) = A_\mu(x).$$

- Applications in lattice gauge theory ($\sim$ 490 citations of Lüscher's original paper)
  - Topological charge
  - Scale setting
  - Non-perturbative gauge coupling constant
  - Chiral condensate
  - Energy–momentum tensor
  - Fermion current and density operators
  - Supersymmetric theory (Kikuchi–Onogi, Kadoh–Ukita)
  - Chiral gauge theory (Grabowska–Kaplan)
  - Supersymmetric current (Hieda–Kasai–Maskino–Morikawa–H.S.)
  - ...
- The most important property of the gradient flow, which makes these applications possible, is its simple renormalization property.

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

where the term with α_0 is introduced to suppress gauge modes.

- This equation can be formally solved as

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

by using the heat kernel,

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right].$$

- R is the non-linear terms

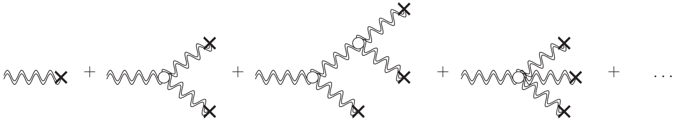
$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]].$$

Perturbative expansion of the gradient flow

- The solution

$$B_\mu(t,x)=\int d^Dy\left[K_t(x-y)_{\mu\nu}A_\nu(y)+\int_0^t ds\,K_{t-s}(x-y)_{\mu\nu}R_\nu(s,y)\right],$$

is represented pictorially as (double lines: K, crosses: A_μ, white circles: R),



Justification of the "gauge fixing term"

- Under the infinitesimal gauge transformation

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x).$$

- Choosing $\omega(t,x)$ as

$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = -\delta\alpha_0 \partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0,$$

α_0 can be changed accordingly

$$\alpha_0 \to \alpha_0 + \delta\alpha_0.$$

- Thus, any gauge invariant quantity (in usual 4D sense) is independent of α_0, as far as it do not contain the flow time derivative ∂_t.

Quantum correlation functions

- Quantum correlation function of the flowed gauge field is obtained by the functional integral over the initial value $A_\mu(x)$:

$$\langle B_{\mu_1}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}A_\mu\, B_{\mu_1}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\, e^{-S_{\rm YM}-S_{\rm gf}-S_{c\bar{c}}}.$$

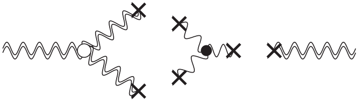
- For example, the contraction of two A_μ's

produces the free propagator of the flowed field

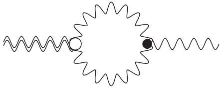
$$\left\langle B^a_\mu(t,x)B^b_\nu(s,y)\right\rangle_0 = \delta^{ab}g_0^2\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2-p_\mu p_\nu)e^{-(t+s)p^2}+\frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].$$

Quantum correlation functions

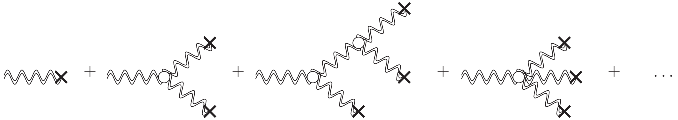
- Similarly, for (black circle: Yang–Mills vertex)



we have the loop flow-line Feynman diagram



- Recall that the flowed gauge field is represented as



Renormalizability of the gradient flow I (Lüscher–Weisz (2011))

- Correlation function of the flowed gauge field

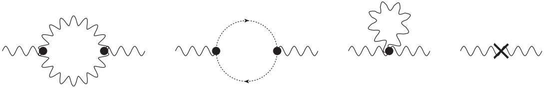
$$\langle B_{\mu_1}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\rangle\,,\qquad t_1>0,\ldots,t_n>0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.

- Two-point function in the tree level (in the Feynman gauge $\lambda_0=\alpha_0=1$)

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0=\delta^{ab}g_0^2\delta_{\mu\nu}\int_p e^{ip(x-y)}\frac{e^{-(t+s)p^2}}{p^2}.$$

- One-loop corrections (consisting only from Yang–Mills vertices)

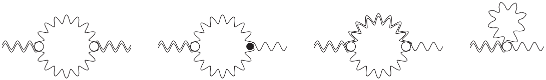


where the last counter term arises from the parameter renormalization

$$g_0^2=\mu^{2\varepsilon}g^2Z,\qquad \lambda_0=\lambda Z_3^{-1}.$$

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)



It turns out that these provide the same effect as the wave function renormalization!

- All order proof of this fact, using a local $D+1$-dimensional field theory, consists the main part of the following lectures.
- No bulk ($t > 0$) counterterm: because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator.
- No boundary ($t = 0$) counterterm besides Yang–Mills ones: because of a BRS symmetry.

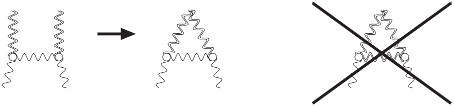
Renormalizability of the gradient flow II

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle \,, \qquad t_1 > 0, \ldots, t_n > 0,$$

remains finite even for the equal-point product

$$t_1 \to t_2, \qquad x_1 \to x_2.$$



- The new loop always contains the gaussian damping factor $\sim e^{-tp^2}$ which makes integral finite; no new UV divergences arise.

Renormalizability of the gradient flow II

- Any composite operators of the flowed gauge field $B_\mu(t,x)$ are automatically renormalized UV finite quantities, although the flowed field is a certain combination of the bare gauge field.
- Such UV finite quantities must be independent of the regularization.
- $\Rightarrow$ E.g., construction of the energy–momentum tensor in lattice gauge theory.

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t\chi(t,x) = [\Delta - \alpha_0\partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
$$\partial_t\bar\chi(t,x) = \bar\chi(t,x)\left[\overleftarrow{\Delta} + \alpha_0\partial_\mu B_\mu(t,x)\right], \qquad \bar\chi(t=0,x) = \bar\psi(x),$$

where

$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$
$$\overleftarrow{\Delta} = \overleftarrow{D}_\mu\overleftarrow{D}_\mu, \qquad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - B_\mu.$$

- It turns out that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar\chi_R(t,x) = Z_\chi^{1/2}\bar\chi(t,x),$$
$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2}C_F 3\frac{1}{\varepsilon} + O(g^4).$$

- Still, any composite operators of $\chi_R(t,x)$ are UV finite.

A $D+1$-dimensional local field theory

- The total action

$$S_{\text{tot}} = S + S_{\text{gf}} + S_{c\bar{c}} + S_{\text{fl}} + S_{d\bar{d}}.$$

- The 4-dimensional part (pure Yang–Mills):

$$S = -\frac{1}{2g_0^2}\int d^D x\,\operatorname{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right],$$

$$S_{\text{gf}} + S_{c\bar{c}} = \delta\frac{-2}{g_0^2}\int d^D x\,\operatorname{tr}\left\{\bar{c}(x)\left[\partial_\mu A_\mu(x) - \frac{1}{2\lambda_0}B(x)\right]\right\},$$

where δ is the nilpotent BRS transformation

$$\delta A_\mu(x) = D_\mu c(x), \qquad \delta c(x) = -c(x)^2,$$
$$\delta\bar{c}(x) = B(x), \qquad \delta B(x) = 0.$$

- So

$$\delta S = 0, \qquad \delta(S_{\text{gf}} + S_{c\bar{c}}) = 0.$$

A $D+1$-dimensional local field theory

- The $D+1$-dimensional part

$$S_{\text{fl}} = -2\int_0^\infty dt \int d^Dx\ \text{tr}\left\{L_\mu(t,x)\left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu\right](t,x)\right\}$$

- Integration over the Lagrange multiplier:

$$L_\mu(t,x) = L_\mu^a(t,x)T^a$$

  imposes the flow equation

- Bulk ghost fields

$$S_{d\bar{d}} = -2\int_0^\infty dt \int d^Dx\ \text{tr}\left\{\bar{d}(t,x)\left[\partial_t d - \alpha_0 D_\mu \partial_\mu d\right](t,x)\right\}$$

- Boundary conditions

$$B_\mu(t=0,x) = A_\mu(x), \qquad d(t=0,x) = c(x),$$

  while $L_\mu(t=0,x)$ and $\bar{d}(t=0,x)$ are integrated freely.

Discrete flow time prescription

- We have to make the meaning precise:

$$\int_0^\infty dt \to \epsilon \sum_{t=0}^{\infty}, \qquad \partial_t B_\mu(t,x) \to \frac{1}{\epsilon}\left[B_\mu(t+\epsilon,x) - B_\mu(t,x)\right]$$

- In particular,

$$\partial_t B_\mu(t=0,x) = \frac{1}{\epsilon}\left[B_\mu(t=\epsilon,x) - A_\mu(x)\right],$$
$$\partial_t d(t=0,x) = \frac{1}{\epsilon}\left[d(t=\epsilon,x) - c(x)\right].$$

Discrete flow time prescription

- Heat kernel with the discretized flow time:

$$K_t^\epsilon(x)_{\mu\nu} \equiv \int_p \frac{e^{ipx}}{p^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{t/\epsilon} + p_\mu p_\nu(1-\epsilon\alpha_0 p^2)^{t/\epsilon}\right].$$

This fulfills

$$\frac{1}{\epsilon}\left[K_{t+\epsilon}^\epsilon(x)_{\mu\nu} - K_t^\epsilon(x)_{\mu\nu}\right] = \partial_\rho\partial_\rho K_t^\epsilon(x)_{\mu\nu} + (\alpha_0 - 1)\partial_\mu\partial_\rho K_t^\epsilon(x)_{\rho\nu},$$

$$K_0^\epsilon(x)_{\mu\nu} = \delta_{\mu\nu}\delta^D(x).$$

- We thus change the variable from B_μ to b_μ by

$$B_\mu(t,x) = \int d^Dy\, K_t^\epsilon(x-y)_{\mu\nu}A_\nu(y) + b_\mu(t,x),$$

the boundary condition then becomes

$$b_\mu(t=0,x) = 0,$$

Discrete flow time prescription

- and the action becomes

$$S_{\mathrm{fl}} = -2\epsilon \sum_{t=0}^{\infty} \int d^D x \,\mathrm{tr}\, L_\mu(t,x) \times \left\{ \frac{1}{\epsilon} \left[b_\mu(t+\epsilon,x) - b_\mu(t,x) \right] - \partial_\nu \partial_\nu b_\mu(t,x) + (1-\alpha_0)\partial_\mu \partial_\nu b_\nu(t,x) \right\} + (\text{interactions}),$$

with the boundary condition

$$b_\mu(t=0,x) = 0.$$

- It is then straightforward to obtain the tree-level bL propagator:

$$\left\langle b_\mu^a(t,x) L_\nu^b(s,y) \right\rangle_0 = \delta^{ab} \vartheta(t-s) K_{t-s-\epsilon}^{\epsilon}(x-y)_{\mu\nu},$$

where $\vartheta(t)$ is a "regularized step function",

$$\vartheta(t) \equiv \begin{cases} 1, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \\ 0, & \text{for } t < 0. \end{cases}$$

Addendum: Derivation of the propagator

- Kinetic term (omitting the spacetime coordinates)

$$S_{\rm fl}^{(2)} = \epsilon \sum_{t=0}^{\infty} L_\mu^a(t) \left\{ \frac{1}{\epsilon} \left[b_\mu^a(t+\epsilon) - b_\mu^a(t) \right] - \partial_\nu \partial_\nu b_\mu^a(t) + (1-\alpha_0) \partial_\mu \partial_\nu b_\nu^a(t) \right\}.$$

- Schwinger–Dyson equation

$$\left\langle \frac{1}{\epsilon} \left[b_\mu^a(t+\epsilon) - b_\mu^a(t) \right] L_\nu^b(s) + \left[-\delta_{\mu\rho} \partial_\sigma \partial_\sigma + (1-\alpha_0) \partial_\mu \partial_\rho \right] b_\rho^a(t) L_\nu^b(s) \right\rangle_0$$
$$= \frac{1}{\epsilon} \delta^{ab} \delta_{\mu\nu} \delta_{t,s},$$

with the boundary condition

$$b_\mu(t=0) = 0.$$

Addendum: Derivation of the propagator

- Schwinger–Dyson equation ($b_\mu(t=0)=0$)

$$\left\langle b_\mu^a(t+\epsilon)L_\nu^b(s)\right\rangle_0 = \{\delta_{\mu\rho}+\epsilon\left[\delta_{\mu\rho}\partial_\sigma\partial_\sigma-(1-\alpha_0)\partial_\mu\partial_\rho\right]\}\left\langle b_\rho^a(t)L_\nu^b(s)\right\rangle_0 + \delta^{ab}\delta_{\mu\nu}\delta_{t,s}.$$

- Step by step solution:

$$\begin{aligned}
&t=0, && \left\langle b_\mu^a(0)L_\nu^b(s)\right\rangle_0=0, && \delta_{0,s}=0,\\
&t=\epsilon, && \left\langle b_\mu^a(\epsilon)L_\nu^b(s)\right\rangle_0=0, && \delta_{\epsilon,s}=0,\\
&t=2\epsilon, && \left\langle b_\mu^a(2\epsilon)L_\nu^b(s)\right\rangle_0=0, && \delta_{2\epsilon,s}=0,\\
&t=\cdots && \cdots && \\
&t=s-\epsilon, && \left\langle b_\mu^a(s-\epsilon)L_\nu^b(s)\right\rangle_0=0, && \delta_{s-\epsilon,s}=0,\\
&t=s, && \left\langle b_\mu^a(s)L_\nu^b(s)\right\rangle_0=0, && \delta_{s,s}=1,\\
&t=s+\epsilon, && \left\langle b_\mu^a(s+\epsilon)L_\nu^b(s)\right\rangle_0=\delta^{ab}\delta_{\mu\nu}, && \delta_{s+\epsilon,s}=0,\\
&t=\cdots && \cdots &&
\end{aligned}$$

Discrete flow time prescription

- Since the AL and bb propagators vanish,

$$\left\langle B_\mu^a(t,x)L_\nu^b(s,y)\right\rangle_0 = \delta^{ab}\vartheta(t-s)K_{t-s-\epsilon}^{\epsilon}(x-y)_{\mu\nu},$$

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}$$

$$\times\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{(t+s)/\epsilon} + \frac{1}{\lambda_0}p_\mu p_\nu(1-\epsilon\alpha_0 p^2)^{(t+s)/\epsilon}\right].$$

- The former BL-propagator becomes the heat kernel (or flow line) for $\epsilon \to 0$.
- We note that, in the present prescription,

$$\left\langle B_\mu^a(t,x)L_\nu^b(t,y)\right\rangle_0 = 0,$$

  and thus there is no loop diagram consisting only of the heat kernels (or flow lines).
- Similar remark applies also to $d\bar{d}$ system.

BRS invariance of the $D+1$-dimensional system

- Going back to the full system...
- The $D+1$-dimensional BRS transformation

$$\delta B_\mu(t,x) = D_\mu d(t,x), \qquad \delta d(t,x) = -d(t,x)^2,$$
$$\delta L_\mu(t,x) = [L_\mu, d](t,x), \qquad \delta \bar{d}(t,x) = D_\mu L_\mu(t,x) - \{d, \bar{d}\}(t,x).$$

This is nilpotent, $\delta^2 = 0$.

BRS invariance of the $D+1$-dimensional system

- BRS invariance of the $D+1$-dimensional part:

$$S_{\text{fl}} + S_{d\bar{d}} = -2\int_0^\infty dt \int d^Dx\ \text{tr}\left[L_\mu(t,x)E_\mu(t,x) + \bar{d}(t,x)e(t,x)\right],$$

where

$$E_\mu(t,x) \equiv \partial_t B_\mu(t,x) - D_\nu G_{\nu\mu}(t,x) - \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$
$$e(t,x) \equiv \partial_t d(t,x) - \alpha_0 D_\mu \partial_\mu d(t,x).$$

- Under the BRS transformation,

$$\delta E_\mu(t,x) = [E_\mu, d](t,x) + D_\mu e(t,x),$$
$$\delta e(t,x) = -\{e, d\}(t,x),$$

and

$$\delta\left(S_{\text{fl}} + S_{d\bar{d}}\right) = 0.$$

BRS invariance of the $D+1$-dimensional system

- Caveat: The fact is that for $\epsilon \neq 0$ the BRS symmetry is broken by $O(\epsilon)$ terms...
- It turns out that those breaking terms are harmless.
- For details, see, Kenji Hieda, Hiroki Makino, H. S., “Proof of the renormalizability of the gradient flow”, arXiv:1604.06200 [hep-lat].

Ward–Takahashi relation (or the Zinn-Justin equation)

- To derive the Ward–Takahashi relation, we introduce the source term

$$S_J = 2\int d^Dx\ \mathrm{tr}\left[J_\mu^A(x)A_\mu(x) + J^c(x)c(x) + J^{\bar{c}}(x)\bar{c}(x) + J^B(x)B(x)\right]$$
$$+ 2\,⨍_0^\infty dt \int d^Dx\ \mathrm{tr}\left[J_\mu^B(t,x)B_\mu(t,x) + J^d(t,x)d(t,x)\right]$$
$$+ 2\int_0^\infty dt \int d^Dx\ \mathrm{tr}\left[J_\mu^L(t,x)L_\mu(t,x) + J^{\bar{d}}(t,x)\bar{d}(t,x)\right],$$

where

$$⨍_0^\infty dt \equiv \epsilon \sum_{t=\epsilon}^{\infty}.$$

Ward–Takahashi relation

- … and

$$
\begin{aligned}
S_K = 2\int d^Dx\ \mathrm{tr}\left[K_\mu^A(x)D_\mu c(x) - K^c(x)c(x)^2\right] \\
+ 2\,⨍_0^\infty dt \int d^Dx\ \mathrm{tr}\left[K_\mu^B(t,x)D_\mu d(t,x) - K^d(t,x)d(t,x)^2\right] \\
+ 2\int_0^\infty dt \int d^Dx\ \mathrm{tr}\Big[K_\mu^L(t,x)[L_\mu, d](t,x) \\
+ K^{\bar{d}}(t,x)\left(D_\mu L_\mu - \{d,\bar{d}\}\right)(t,x)\Big],
\end{aligned}
$$

where

$$
⨍_0^\infty dt \equiv \epsilon \sum_{t=\epsilon}^{\infty}.
$$

Ward–Takahashi relation

- Considering the BRS transformation of integration variables,

$$\left\langle -2\int d^Dx\,\mathrm{tr}\left[J^A_\mu(x)D_\mu c(x)+J^c(x)c(x)^2-J^{\bar{c}}(x)B(x)\right]\right\rangle$$
$$+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\left[J^B_\mu(t,x)D_\mu d(t,x)+J^d(t,x)d(t,x)^2\right]\right\rangle$$
$$+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\Big[J^L_\mu(t,x)[L_\mu,d](t,x)$$
$$-J^{\bar{d}}\left(D_\mu L_\mu-\{d,\bar{d}\}\right)(t,x)\Big]\right\rangle=0.$$

- Introducing the effective action $\mathit{\Gamma}$,

$$\int d\varphi\, e^{-S+J\varphi+K\delta\varphi}=e^{-W[J,K]},\qquad \frac{\delta}{\delta J}W[J,K]=-\langle\varphi\rangle=-\phi,$$
$$\mathit{\Gamma}[\phi,K]=W[J,K]+J\phi,\quad \frac{\delta}{\delta\phi}\mathit{\Gamma}[\phi,K]=\pm J,\quad \frac{\delta}{\delta K}\mathit{\Gamma}[\phi,K]=\langle\delta\varphi\rangle\,.$$

Ward–Takahashi relation

- The WT relation reads

$$\int d^D x \left[\frac{\delta\Gamma}{\delta A_\mu^a(x)}\frac{\delta\Gamma}{\delta K_\mu^{Aa}(x)}+\frac{\delta\Gamma}{\delta c^a(x)}\frac{\delta\Gamma}{\delta K^{ca}(x)}-\frac{\delta\Gamma}{\delta\bar{c}^a(x)}B^a(x)\right]$$
$$+\int_0^\infty dt\int d^D x\left[\frac{\delta\Gamma}{\delta B_\mu^a(t,x)}\frac{\delta\Gamma}{\delta K_\mu^{Ba}(t,x)}+\frac{\delta\Gamma}{\delta d^a(t,x)}\frac{\delta\Gamma}{\delta K^{da}(t,x)}\right]$$
$$+\int_0^\infty dt\int d^D x\left[\frac{\delta\Gamma}{\delta L_\mu^a(t,x)}\frac{\delta\Gamma}{\delta K_\mu^{La}(t,x)}+\frac{\delta\Gamma}{\delta\bar{d}^a(t,x)}\frac{\delta\Gamma}{\delta K^{\bar{d}a}(t,x)}\right]$$
$$=0.$$

Equation of motion of $B(x)$ and $\bar{c}(x)$

- Equation of motion:

$$\left\langle \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right]\right\rangle - J^{Ba}(x) = 0,$$

$$\left\langle -\frac{1}{g_0^2}\partial_\mu D_\mu c^a(x)\right\rangle + J^{\bar{c}a}(x) = 0.$$

- In terms of the effective action,

$$\frac{\delta\Gamma}{\delta B^a(x)} = \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right],$$

$$\frac{\delta\Gamma}{\delta\bar{c}^a(x)} - \frac{1}{g_0^2}\partial_\mu\frac{\delta\Gamma}{\delta K_\mu^{Aa}(x)} = 0.$$

- These show, defining $\tilde{K}_\mu^{Aa} \equiv K_\mu^{Aa} - \frac{1}{g_0^2}\partial_\mu\bar{c}^a$,

$$\Gamma = \tilde{\Gamma}[\not{B}, \tilde{K}^A, \not{\bar{c}}] - \frac{2}{g_0^2}\int d^Dx\, \mathrm{tr}\left[B(x)\left(\partial_\mu A_\mu - \frac{1}{2\lambda_0}B\right)(x)\right].$$

Reduced effective action

- The reduced effective action

$$\tilde{\Gamma} = \tilde{\Gamma}[A_\mu, c, \tilde{K}^A_\mu, K^c; B_\mu, d, L_\mu, \bar{d}, K^B_\mu, K^d, K^L_\mu, K^{\bar{d}}]$$

$$\begin{aligned}
&\int d^Dx \left[\frac{\delta\tilde{\Gamma}}{\delta A^a_\mu(x)} \frac{\delta\tilde{\Gamma}}{\delta \tilde{K}^{Aa}_\mu(x)} + \frac{\delta\tilde{\Gamma}}{\delta c^a(x)} \frac{\delta\tilde{\Gamma}}{\delta K^{ca}(x)} \right] \\
&\quad + \int_0^\infty dt \int d^Dx \left[\frac{\delta\tilde{\Gamma}}{\delta B^a_\mu(t,x)} \frac{\delta\tilde{\Gamma}}{\delta K^{Ba}_\mu(t,x)} + \frac{\delta\tilde{\Gamma}}{\delta d^a(t,x)} \frac{\delta\tilde{\Gamma}}{\delta K^{da}(t,x)} \right] \\
&\qquad + \int_0^\infty dt \int d^Dx \left[\frac{\delta\tilde{\Gamma}}{\delta L^a_\mu(t,x)} \frac{\delta\tilde{\Gamma}}{\delta K^{La}_\mu(t,x)} + \frac{\delta\tilde{\Gamma}}{\delta \bar{d}^a(t,x)} \frac{\delta\tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)} \right] \\
&\qquad\quad = 0.
\end{aligned}$$

Renormalization constants

- Renormalization ($D = 4 - 2\varepsilon$)

$$g_0^2 = \mu^{2\varepsilon} g^2 Z,$$
$$A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a, \qquad \tilde{K}_\mu^{Aa} = Z^{-1/2} Z_3^{-1/2} (\tilde{K}_R^A)_\mu^a,$$
$$c^a = \tilde{Z}_3 Z^{1/2} Z_3^{1/2} c_R^a, \qquad K^{ca} = \tilde{Z}_3^{-1} Z^{-1/2} Z_3^{-1/2} K_R^{ca},$$

and

$$\lambda_0 = \lambda Z_3^{-1},$$
$$B^a = Z^{1/2} Z_3^{-1/2} B_R^a, \qquad \bar{c}^a = Z^{1/2} Z_3^{-1/2} \bar{c}_R^a.$$

- Note: $(\tilde{K}_R^A)_\mu^a = (K_R^A)_\mu^a - \frac{1}{\mu^{2\varepsilon} g^2} \partial_\mu \bar{c}_R^a$, and

$$-\frac{2}{g_0^2} \operatorname{tr} \left[B \left(\partial_\mu A_\mu - \frac{1}{2\lambda_0} B \right) \right] = -\frac{2}{\mu^{2\varepsilon} g^2} \operatorname{tr} \left\{ B_R \left[\partial_\mu (A_R)_\mu - \frac{1}{2\lambda} B_R \right] \right\}$$

- We want to show that the above renormalization is enough to make $\tilde{\Gamma}$ finite!

WT relation

- WT relation in terms of renormalized quantity

$$
\begin{aligned}
&\int d^D x \left[\frac{\delta \tilde{\Gamma}}{\delta (A_R)^a_\mu(x)} \frac{\delta \tilde{\Gamma}}{\delta (\tilde{K}^A_R)^a_\mu(x)} + \frac{\delta \tilde{\Gamma}}{\delta c^a_R(x)} \frac{\delta \tilde{\Gamma}}{\delta K^{ca}_R(x)} \right] \\
&\quad + \int_0^\infty dt \int d^D x \left[\frac{\delta \tilde{\Gamma}}{\delta B^a_\mu(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K^{Ba}_\mu(t,x)} + \frac{\delta \tilde{\Gamma}}{\delta d^a(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K^{da}(t,x)} \right] \\
&\qquad + \int_0^\infty dt \int d^D x \left[\frac{\delta \tilde{\Gamma}}{\delta L^a_\mu(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K^{La}_\mu(t,x)} + \frac{\delta \tilde{\Gamma}}{\delta \bar{d}^a(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)} \right] \\
&\qquad\quad = 0.
\end{aligned}
$$

- Loop expansion in the renormalized perturbation theory

$$
\tilde{\Gamma} = \sum_{\ell=0}^\infty \tilde{\Gamma}^{(\ell)}
$$

- Tree-level effective action in the renormalized perturbation theory

$$
\tilde{\Gamma}^{(0)} = (S + S_{\mathrm{fl}} + S_{d\bar{d}} + S_K)\big|_{Z=Z_3=\tilde{Z}_3=1, K^A_\mu \to (\tilde{K}^A_R)_\mu}
$$

Renormalizability of the gradient flow

- We want to show that the renormalization constants

$$Z, \qquad Z_3, \qquad \tilde{Z}_3,$$

can be chosen order by order in the loop expansion, such as $\tilde{\Gamma}$ is finite.

- The corresponding counter term is given by

$$\Delta S = S + S_{\text{fl}} + S_{d\bar{d}} + S_K - (S + S_{\text{fl}} + S_{d\bar{d}} + S_K)|_{Z=Z_3=\tilde{Z}_3=1, K_\mu^A \to (\tilde{K}_R^A)_\mu} .$$

- This contains in particular the “boundary counter term”

$$\Delta S_{\text{bc}} \equiv 2 \int d^D x \, \text{tr} \left[L_\mu(0,x)(Z^{1/2} Z_3^{1/2} - 1)(A_R)_\mu(x) \right] + 2 \int d^D x \, \text{tr} \left[\bar{d}(0,x)(\tilde{Z}_3^{1/2} Z^{1/2} Z_3^{1/2} - 1) c_R(x) \right] .$$

- In the renormalized perturbation theory, the boundary conditions are taken as

$$B_\mu(t=0,x) = (A_R)_\mu(x), \qquad d(t=0,x) = c_R(x).$$

Renormalizability of the gradient flow

- Mathematical induction: Assume that, to the ℓ-th order Z, Z_3, and $\tilde{Z}_3$ can be chosen so that $\tilde{\Gamma}^{(\ell)}$ is finite.
- Then, writing the divergent part of $\tilde{\Gamma}^{(\ell+1)}$, $\tilde{\Gamma}^{(\ell+1)\text{div}}$, it satisfies the WT relation

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\text{div}} = 0,$$

where

$$\begin{aligned}
\tilde{\Gamma}^{(0)} * \equiv &-\int d^D x \left[\frac{\delta \tilde{\Gamma}^{(0)}}{\delta (A_R)^a_\mu(x)} \frac{\delta}{\delta (\tilde{K}^A_R)^a_\mu(x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta (\tilde{K}^A_R)^a_\mu(x)} \frac{\delta}{\delta (A_R)^a_\mu(x)} \right. \\
&\qquad \left. + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta c^a_R(x)} \frac{\delta}{\delta K^{ca}_R(x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{ca}_R(x)} \frac{\delta}{\delta c^a_R(x)} \right] \\
&- \epsilon \int d^D x \left[\frac{\delta \tilde{\Gamma}^{(0)}}{\delta L^a_\mu(0,x)} \frac{\delta}{\delta K^{La}_\mu(0,x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{La}_\mu(0,x)} \frac{\delta}{\delta L^a_\mu(0,x)} \right. \\
&\qquad \left. + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \bar{d}^a(0,x)} \frac{\delta}{\delta K^{\bar{d}a}(0,x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{\bar{d}a}(0,x)} \frac{\delta}{\delta \bar{d}^a(0,x)} \right] \\
&+ (\text{derivatives w.r.t. } t \neq 0 \text{ variables})
\end{aligned}$$

Renormalizability of the gradient flow

- Nilpotency:

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(0)} * = 0.$$

- In the equation,

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}} = 0,$$

we decompose

$$\tilde{\Gamma}^{(0)} * = \tilde{\Gamma}^{(0)}_{4\mathrm{D}} * + \tilde{\Gamma}^{(0)}_{5\mathrm{D}} *,$$

where $\tilde{\Gamma}^{(0)}_{4\mathrm{D}} *$ is the corresponding operator in the original 4D gauge theory:

$$\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \equiv -\int d^D x \left[\frac{\delta \left.\tilde{\Gamma}^{(0)}\right|_{L_\mu=\bar{d}=0}}{\delta (A_R)^a_\mu(x)} \frac{\delta}{\delta (\tilde{K}^A_R)^a_\mu(x)} + \frac{\delta \left.\tilde{\Gamma}^{(0)}\right|_{L_\mu=\bar{d}=0}}{\delta (\tilde{K}^A_R)^a_\mu(x)} \frac{\delta}{\delta (A_R)^a_\mu(x)} + \frac{\delta \left.\tilde{\Gamma}^{(0)}\right|_{L_\mu=\bar{d}=0}}{\delta c^a_R(x)} \frac{\delta}{\delta K^{ca}_R(x)} + \frac{\delta \left.\tilde{\Gamma}^{(0)}\right|_{L_\mu=\bar{d}=0}}{\delta K^{ca}_R(x)} \frac{\delta}{\delta c^a_R(x)} \right],$$

and

$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{4\mathrm{D}}(\not{L}_\mu, \not{\bar{d}}, \dots) + \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu, \bar{d}, \dots)$$

Renormalizability of the gradient flow

- Then

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\text{div}} = 0,$$

is decomposed into

$$\tilde{\Gamma}^{(0)}_{4D} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{4D} = 0,$$
$$\tilde{\Gamma}^{(0)}_{4D} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{5D} + \tilde{\Gamma}^{(0)}_{5D} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{4D} + \tilde{\Gamma}^{(0)}_{5D} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{5D} = 0.$$

- The general solution to the first equation is known to be

$$\tilde{\Gamma}^{(\ell+1)\text{div}}_{4D} = -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\, \text{tr}\left[x_1 (F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] - 2\tilde{\Gamma}^{(0)}_{4D} * \int d^Dx\, \text{tr}\, \underbrace{\left[y_1 (\tilde{K}^A_R)_\mu(x)(A_R)_\mu(x) + y_2 K^c_R(x) c_R(x)\right]}_{\text{dim.}=3,\text{ghost number}=-1},$$

where

$$(F_R)_{\mu\nu}(x) \equiv \partial_\mu (A_R)_\nu(x) - \partial_\nu (A_R)_\mu(x) + [(A_R)_\mu, (A_R)_\nu](x).$$

Renormalizability of the gradient flow

- Thus, we have

$$
\begin{aligned}
\tilde{\Gamma}^{(\ell+1)\mathrm{div}} &= -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\,\mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] \\
&\qquad - 2\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \int d^Dx\,\mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2K_R^c(x)c_R(x)\right] \\
&\qquad\qquad + \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu,\bar{d},\dots) \\
&= -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\,\mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] \\
&\qquad - 2\tilde{\Gamma}^{(0)} * \int d^Dx\,\mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2K_R^c(x)c_R(x)\right] \\
&\qquad\qquad + \tilde{\tilde{\Gamma}}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu,\bar{d},\dots),
\end{aligned}
$$

where

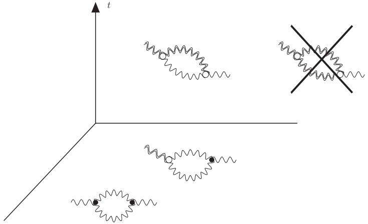
$$
\tilde{\tilde{\Gamma}}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} \equiv \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} + 2\int d^Dx\,\mathrm{tr}\left[y_1L_\mu(0,x)(A_R)_\mu(x) - y_2\bar{d}(0,x)c_R(x)\right].
$$

Renormalizability of the gradient flow

- Possible divergent terms are boundary ones:

$$\tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}} = -2\int d^D x \,\operatorname{tr}\left[z_1 L_\mu(0,x)(A_R)_\mu(x) + z_2 \bar{d}(0,x) c_R(x)\right].$$

This comes from the consideration such as



- Then the WT relation yields,

$$\tilde{\Gamma}^{(0)} * \tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}}$$
$$= -2\int d^D x \,\operatorname{tr}[(z_1 - z_2) L_\mu(0,x)\partial_\mu c_R(x)$$
$$- z_2 L_\mu(0,x)[(A_R)_\mu, c_R](x) - z_2 \bar{d}(0,x) c_R(x)^2] = 0.$$

Renormalizability of the gradient flow

- This shows that

$$\tilde{\tilde{\Gamma}}_{5\text{D}}^{(\ell+1)\text{div}} = 0,$$

and

$$\tilde{\Gamma}^{(\ell+1)\text{div}} = -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\ \text{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] - 2\tilde{\Gamma}^{(0)} * \int d^Dx\ \text{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2K_R^c(x)c_R(x)\right].$$

Renormalizability of the gradient flow

- One can confirm that the above possible $\ell+1$-th order divergent part,

$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\ \mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] - 2\tilde{\Gamma}^{(0)} * \int d^Dx\ \mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2K_R^c(x)c_R(x)\right],$$

  can be canceled by choosing in the $\ell+1$-th order,

$$Z^{(\ell+1)} = x_1,$$
$$Z_3^{(\ell+1)} = -x_1 + 2y_1,$$
$$\tilde{Z}_3^{(\ell+1)} = -y_1 - y_2$$

  in

$$\left.(S + S_{\mathrm{fl}} + S_{d\bar{d}} + S_K)\right|_{K_\mu^A \to (\tilde{K}_R^A)_\mu}.$$

- This completes the proof that no wave function renormalization of the flowed gauge field is required.

Addendum: Why K_μ^L and $K^{\bar{d}}$ do not appear in the divergent term

- First, we have to see that

$$-2\int d^Dx\ \mathrm{tr}\left[\partial_t B_\mu(0,x)(A_R)_\mu(x)+\partial_t d(0,x)\bar{c}_R(x)\right],$$

  does not appear in the boundary divergence, although the mass dimension and the ghost number perfectly match.
- This follows from the fact that there is no loop being consist only of the BL (or $d\bar{d}$) propagator. Divergent term containing B (or d) must accomplish L ($\bar{d}$).
- Next we note in $\tilde{\Gamma}^{(0)}*$,

$$\frac{\delta\tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0,x)}\sim\partial_t B_\mu^a(0,x)+\cdots,\qquad\frac{\delta\tilde{\Gamma}^{(0)}}{\delta\bar{d}^a(0,x)}\sim\partial_t d^a(0,x)+\cdots,$$

  and

$$\tilde{\Gamma}^{(0)}*\sim\frac{\delta\tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0,x)}\frac{\delta}{\delta K_\mu^{La}(0,x)}+\frac{\delta\tilde{\Gamma}^{(0)}}{\delta\bar{d}^a(0,x)}\frac{\delta}{\delta K^{\bar{d}a}(0,x)}+\cdots.$$

Fermion (or general matter) fields

- How the argument modified when fermion (or general matter) fields are included?

$$\cdots + \int d^D x\, \bar{\psi}(x)(\not{D} + m_0)\psi(x)$$

$$\cdots + \int_0^\infty dt \int d^D x \left[\bar{\lambda}(t,x)(\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu)\chi(t,x) + \bar{\chi}(t,x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\nu B_\nu)\lambda(t,x)\right].$$

- Boundary conditions

$$\chi(t=0,x) = \psi(x), \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x).$$

- BRS transformation

$$\delta\psi(x) = -c(x)\psi(x), \qquad \delta\bar{\psi}(x) = -\bar{\psi}(x)c(x)$$

$$\delta\chi(t,x) = -d(t,x)\chi(t,x), \qquad \delta\bar{\chi}(t,x) = -\bar{\chi}(t,x)d(t,x)$$

and

$$\delta\lambda(t,x) = -d(t,x)\lambda(t,x), \qquad \delta\bar{\lambda}(t,x) = -\bar{\lambda}(t,x)d(t,x).$$

Fermion (or general matter) fields

- It turns out that the total action can be made BRS invariant by simply modifying the BRS transformation of $\bar{d}(t,x)$ to

$$\delta\bar{d}(t,x) = \cdots + \bar{\lambda}(t,x)T^a\chi(t,x)T^a - \bar{\chi}(t,x)T^a\lambda(t,x)T^a.$$

- The nilpotency $\delta^2 = 0$ is preserved under this modification.

Fermion (or general matter) fields

- General form of the divergent part containing the new fields is

$$
\begin{aligned}
&\tilde{\Gamma}^{(\ell+1)\mathrm{div}} \\
&= \cdots + \int d^Dx \, \left\{ w_1 \bar{\psi}_R(x)\gamma_\mu \left[\partial_\mu + (A_R)_\mu(x)\right] \psi_R(x) + w_2 m_R \bar{\psi}_R(x)\psi_R(x) \right\} \\
&\qquad + \tilde{\Gamma}^{(0)} * \int d^Dx \, w_3 \left[\bar{K}_R^{\psi}(x)\psi_R(x) + \bar{\psi}_R(x) K_R^{\psi}(x) \right] \\
&\qquad\qquad + \int d^Dx \, \xi_1 \left[\bar{\lambda}_R(0,x)\psi_R(x) + \bar{\psi}_R(x)\lambda_R(0,x) \right].
\end{aligned}
$$

- These divergences are canceled by

$$
Z_\psi^{(\ell+1)} = w_1 + 2w_3, \qquad Z_m^{(\ell+1)} = -w_1 + w_2, \qquad Z_\chi^{(\ell+1)} = w_1 + 4w_3 - 2\xi_1,
$$

in $m_0 = Z_m^{-1} m_R$, $\psi = Z_\psi^{-1/2}\psi_R$, and $\bar{\psi} = Z_\psi^{-1/2}\bar{\psi}_R$ and

$$
\begin{aligned}
\bar{\lambda} &= Z_\chi^{1/2}\bar{\lambda}_R, & \lambda &= Z_\chi^{1/2}\lambda_R, \\
\chi &= Z_\chi^{-1/2}\chi_R, & \bar{\chi} &= Z_\chi^{-1/2}\bar{\chi}_R.
\end{aligned}
$$

(Renormalization of Ks is omitted.)

Fermion (or general matter) fields

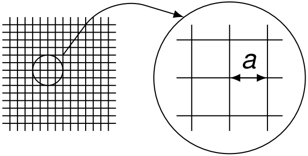
- Thus, the renormalization of flowed fermion fields is not excluded and an explicit calculation shows

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_F 3 \frac{1}{\varepsilon} + O(g^4).$$

- The situation must be similar for generic matter fields.

Lattice gauge theory and the energy–momentum tensor (EMT)

- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime...



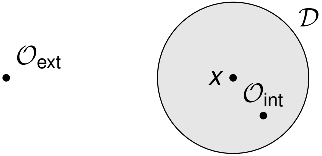
- internal gauge symmetry is preserved exactly...
- but incompatible with spacetime symmetries (translation, Poincaré, SUSY, conformal, ...) for $a \neq 0$.
- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $T_{\mu\nu}(x)$.
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \stackrel{a \to 0}{\to} 1.$$

EMT in lattice gauge theory?

- Is it possible to construct EMT on the lattice, which becomes the correct EMT automatically in the continuum limit $a \to 0$?
- The correct EMT is characterized by the Ward–Takahashi relation

$$\left\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} d^D x \, \partial_\mu T_{\mu\nu}(x) \, \mathcal{O}_{\text{int}} \right\rangle = - \langle \mathcal{O}_{\text{ext}} \, \partial_\nu \mathcal{O}_{\text{int}} \rangle \, .$$



- This contains the correct normalization and the conservation law.

EMT in lattice gauge theory?

- If such a construction is possible, we expect wide application to physics related to spacetime symmetries: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, ...
- Also the present work is an attempt to define EMT in quantum field theory in the non-perturbative level.

Conventional approach (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right) \psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{\not{D}} \psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x) \psi(x),$$

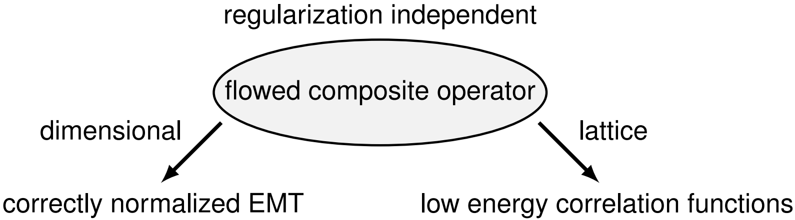
and, Lorentz non-covariant ones:

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

- Seven non-universal coefficients Z_i must be determined by lattice perturbation theory or by a non-perturbative method

Our approach (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool.
- Schematically,



EMT in the dimensional regularization

- The action

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

- Under the localized translation (plus the gauge transformation),

$$\delta A_\mu(x) = \xi_\nu(x)F_{\nu\mu}(x),$$

$$\delta\psi(x) = \xi(x)_\mu D_\mu\psi(x), \qquad \delta\bar{\psi}(x) = \xi(x)_\mu\bar{\psi}(x)\overleftarrow{D}_\mu,$$

we have

$$\delta S = -\int d^Dx\,\xi_\nu(x)\partial_\mu\left[T_{\mu\nu}(x) + A_{\mu\nu}(x)\right],$$

where the anti-symmetric part,

$$A_{\mu\nu}(x) = \frac{1}{4}\bar{\psi}(x)\sigma_{\mu\nu}(\not{D} + m_0)\psi(x) + \frac{1}{4}\bar{\psi}(x)(\overleftarrow{\not{D}} - m_0)\sigma_{\mu\nu}\psi(x),$$

is proportional to the EoM and can be neglected...

EMT in dimensional regularization

- … and $T_{\mu\nu}(x)$ is the symmetric EMT:

$$T_{\mu\nu}(x)$$
$$= \frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\} + \frac{1}{4}\mathcal{O}_{3\mu\nu}(x) - \frac{1}{2}\mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \text{VEV},$$

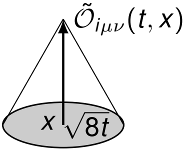
where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu}\sum_{\rho,\sigma} F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),$$
$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),$$
$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x)\psi(x).$$

- We define the renormalized EMT by subtracting its possibly divergent vacuum expectation value.
- Under the dimensional regularization, this is the correct EMT.

Small flow-time expansion

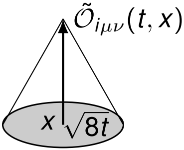
- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.
- But how?
- In general, the relation between composite operators in $t > 0$ (heaven) and in 4D (the earth) is not obvious at all...
- The relation becomes tractable, in the limit in which the flow time becomes small $t \to 0$.
- Small flow-time expansion (Lüscher–Weisz (2011)):



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

Small flow-time expansion

- Small flow-time expansion:



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

- Inverting this relation,

$$\mathcal{O}_{i\mu\nu}(x) - \mathrm{VEV} = \lim_{t\to 0} \left\{ \sum_j \left(\zeta^{-1}\right)_{ij}(t) \left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x) \rangle \mathbb{1} \right] \right\}.$$

- So, if we know the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$, the 4D operator in the LHS can be extracted as the $t \to 0$ limit.

Renormalization group argument

- We are interested in the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ in

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \langle \mathcal{O}_{j\mu\nu}(x) \rangle \mathbb{1} \right] + O(t).$$

- If all the composite operators in this relation are made out from bare quantities,

$$\left(\mu \frac{\partial}{\partial \mu} \right)_0 \zeta_{ij}(t) = 0,$$

  and $\zeta_{ij}(t)$ are indep. of the renormalization scale μ, when expressed in terms of running parameters. We can set $\mu = c/\sqrt{t}$, and

$$\zeta_{ij}(t)\,[g, m; \mu] = \zeta_{ij}(t) \left[g(c/\sqrt{t}), m(c/\sqrt{t}); c/\sqrt{t} \right].$$

- For $t \to 0$, $g(c/\sqrt{8t}) \to 0$ because of the asymptotic freedom; use of perturbation theory is thus justified!

Ringed fermion fields

- Recall that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2}\bar{\chi}(t,x),$$

  although composite operators of $\chi_R(t,x)$ are UV finite.

- To avoid the complication associated with this, we introduce

$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}} = \chi_R(t,x) + O(g^2),$$

  where

$$\mathcal{C} \equiv \sqrt{\frac{-2\dim(R)}{(4\pi)^2}},$$

  and similarly for $\bar{\chi}(t,x)$.

- Since Z_χ is canceled out in $\mathring{\chi}(t,x)$, any composite operators of $\mathring{\chi}(t,x)$ and $\mathring{\bar{\chi}}(t,x)$ are UV finite.

EMT from the gradient flow

- We take following composite operators of flowed fields:

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
$$\tilde{\mathcal{O}}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x),$$
$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

and then set the small flow-time expansion:

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle\tilde{\mathcal{O}}_{i\mu\nu}(t,x)\right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t)\left[\mathcal{O}_{j\mu\nu}(x) - \left\langle\mathcal{O}_{j\mu\nu}(x)\right\rangle \mathbb{1}\right] + O(t).$$

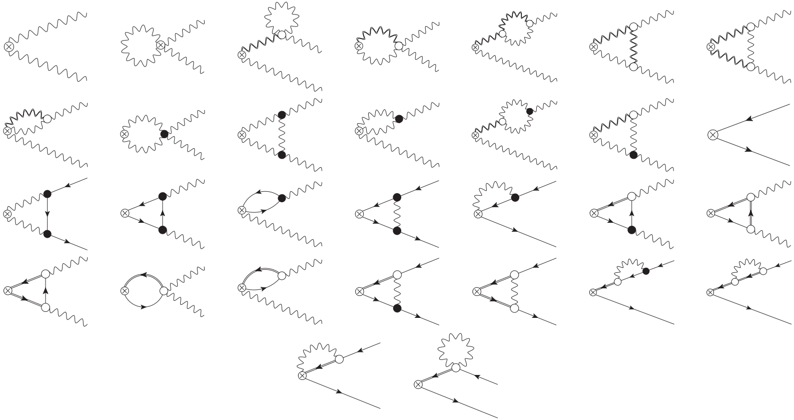
- We compute $\zeta_{ij}(t)$ to the one-loop order and substitute

$$\mathcal{O}_{i\mu\nu}(x) - \left\langle\mathcal{O}_{i\mu\nu}(x)\right\rangle \mathbb{1} = \lim_{t\to 0}\left\{\sum_j \left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle\tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle \mathbb{1}\right]\right\},$$

in the expression of the EMT in the dimensional regularization

Computation of expansion coefficients $\zeta_{ij}(t)$

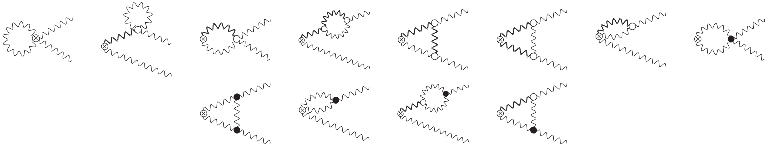
- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



- Even to write down correct set of diagrams is tedious. . .
- . . . and it is very easy to make mistakes in the loop calculation, as I actually did!

A very simple and quick computational scheme (arXiv:1507.02360)

- Let us consider the pure Yang–Mills theory for which the conventional approach would require



- The small flow-time expansion reads

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)$$
$$\overset{t\to 0}{\sim} \left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right\rangle \mathbb{1}$$
$$+ \zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) + O(t).$$

A very simple and quick computational scheme

- The EMT can then be given by

$$T_{\mu\nu} = \lim_{t\to 0}\left\{ c_1(t)\left[G^a_{\mu\rho}G^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right] + c_2(t)\left[\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma} - \left\langle \delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right\rangle \mathbb{1}\right]\right\},$$

where

$$c_1(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[2 - \zeta_{11}(t)\right], \qquad c_2(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[-\frac{1}{2}\varepsilon\zeta_{12}(t)\right].$$

Background field method in the gradient flow

- The “gauge fixing term” (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

leads to

$$\langle B_\mu B_\nu \rangle_0 \sim \frac{1}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].$$

- Here, we adopt instead

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t,x),$$

where we have set

$$B_\mu(t,x) = \underbrace{\hat{B}_\mu(t,x)}_{\text{background}} + \underbrace{b_\mu(t,x)}_{\text{quantum}}$$

and

$$\hat{D}_\mu = \partial_\mu + [\underbrace{\hat{B}_\mu(t,x)}_{\text{background}}, \cdot]$$

is the covariant derivative wrt the background field.

Background field method in the gradient flow

- Our "gauge fixing term" breaks the covariance under the full gauge transformation, but preserves covariance under the background gauge transformation:

$$\hat{B}_\mu \to \hat{B}_\mu + \hat{D}_\mu \omega(x), \qquad b_\mu \to b_\mu + [b_\mu, \omega(x)].$$

- We also postulate that the background field obeys its own flow equation:

$$\partial_t \hat{B}_\mu(t,x) = \hat{D}_\nu \hat{G}_{\nu\mu}(t,x), \qquad \hat{B}_\mu(0,x) = \hat{A}_\mu(x),$$

  and assume furthermore that

$$\hat{D}_\nu \hat{F}_{\nu\mu}(x) = 0,$$

  for simplicity.
- Then the background field does not flow:

$$\hat{B}_\mu(t,x) = \hat{A}_\mu(x).$$

The background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

- Background gauge fixing term

$$S_{\text{gf}} = -\frac{\lambda_0}{g_0^2}\int d^Dx\ \text{tr}\left[\hat{D}_\mu a_\mu(x)\hat{D}_\nu a_\nu(x)\right],$$

and the corresponding Faddeev–Popov ghost action

$$S_{c\bar{c}} = \frac{2}{g_0^2}\int d^Dx\ \text{tr}\left[\bar{c}(x)\hat{D}_\mu D_\mu c(x)\right],$$

preserve invariance under the background gauge transformation

$$\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \hat{D}_\mu\omega(x), \qquad a_\mu(x) \to a_\mu(x) + [a_\mu(x), \omega(x)].$$

- This greatly simplifies perturbative calculations (of renormalization constants, for example).

With our gauge fixing term in the flow equation

- Any gauge invariant quantity (that does not contain ∂_t) is independent of α_0
- Manifestly background gauge covariant expressions
- Tree-level propagator in the presence of the background field (in the "Feynman gauge" $\lambda_0 = \alpha_0 = 1$) is

$$\left\langle b_\mu^a(t,x) b_\nu^b(s,y) \right\rangle_{\hat{A},0} = g_0^2 \left(e^{(t+s)\hat{\Delta}_x} \frac{1}{-\hat{\Delta}_x} \right)_{\mu\nu}^{ab} \delta(x-y),$$

where

$$\hat{\Delta}_{\mu\nu}^{ab} = (\hat{\mathcal{D}}^2)^{ab} \delta_{\mu\nu} + 2\hat{\mathcal{F}}_{\mu\nu}^{ab}, \qquad (\hat{\mathcal{D}}^2)^{ab} = \hat{\mathcal{D}}_\mu^{ac} \hat{\mathcal{D}}_\mu^{cb},$$

and

$$\hat{\mathcal{D}}_\mu^{ab} \equiv \delta^{ab} \partial_\mu + \hat{A}_\mu^c(x) f^{acb}, \qquad \hat{\mathcal{F}}_{\mu\nu}^{ab}(x) \equiv \hat{F}_{\mu\nu}^c(x) f^{acb}.$$

Small flow time expansion by the background field method

- In the small flow time expansion,

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) \overset{t\to 0}{\sim} \mathrm{VEV}\times\mathbb{1} + \zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) + O(t),$$

we substitute the background-quantum decomposition as

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) = \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\big|_{O(b^1)} + G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\big|_{O(b^2)} + O(b^3$$

and

$$F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) = \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\big|_{O(a^1)} + F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\big|_{O(a^2)} + O(a^3),$$

- Then, we take the expectation value in the presence of the background field noting $\langle\mathbb{1}\rangle_{\hat{A}} = 1$.

Small flow time expansion by the background field method

- Then we have

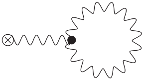
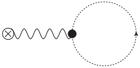
$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^1)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^1)}\right\rangle_{\hat{A}}$$
$$+\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}$$
$$\overset{t\to 0}{\sim} \text{VEV}\times 1 + [\zeta_{11}(t)-1]\,\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) + O(t)$$
$$+\,(\text{2-loop quantities}),$$

where the first line reads

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^1)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^1)}\right\rangle_{\hat{A}}$$
$$=\hat{F}^a_{\mu\rho}\left[\hat{D}_\nu\left\langle b_\rho(t,x)-a_\rho(x)\right\rangle_{\hat{A}} - \hat{D}_\rho\left\langle b_\nu(t,x)-a_\nu(x)\right\rangle_{\hat{A}}\right]^a$$
$$+\left[\hat{D}_\mu\left\langle b_\rho(t,x)-a_\rho(x)\right\rangle_{\hat{A}} - \hat{D}_\rho\left\langle b_\mu(t,x)-a_\mu(x)\right\rangle_{\hat{A}}\right]^a\hat{F}^a_{\nu\rho}.$$

Tadpoles

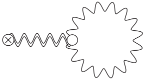
- First type (Yang–Mills vertex only):

These identically vanish (!) under the dimensional regularization for which

$$\int_p \frac{1}{(p^2)^\alpha} = 0.$$

- Second type (flow vertex):



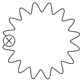
By the background gauge covariance,

$$\left\langle b^a_\mu(t,x)\right\rangle_{\hat{A}} \sim t\hat{D}_\nu \hat{F}^a_{\nu\mu}(x) + O(t^2).$$

This is higher order in t and does not contribute. . .

"1PI" diagram

- The remaining 1-loop diagram is:



Oh! There is only a single diagram!!!

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}$$

$$= 2g_0^2 \int_0^t d\xi \lim_{y\to x} \mathrm{tr}\Big[\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\hat{\mathcal{D}}_\alpha \left(e^{2\xi\hat{\Delta}}\right)_{\beta\gamma}\hat{\mathcal{D}}_\delta$$

$$+ \hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\mu}\Big]\delta(x-y),$$

where $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma} \equiv \delta_{\mu\alpha}\delta_{\nu\delta}\delta_{\beta\gamma} - \delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\beta\delta} - \delta_{\mu\beta}\delta_{\nu\delta}\delta_{\alpha\gamma} + \delta_{\mu\beta}\delta_{\nu\gamma}\delta_{\alpha\delta}$.

“1PI” diagram

- The remaining 1-loop diagram is:

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}$$
$$= 2g_0^2\int_0^t d\xi\,\lim_{y\to x}\mathrm{tr}\Big[\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\hat{\mathcal{D}}_\alpha\left(e^{2\xi\hat{\Delta}}\right)_{\beta\gamma}\hat{\mathcal{D}}_\delta$$
$$+\hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\nu}+\hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\mu}\Big]\delta(x-y).$$

- This is evaluated by setting

$$\delta(x-y)=\int_p e^{ipx}e^{-ipy}$$

and noting

$$\hat{\mathcal{D}}_\mu e^{ipx}=e^{ipx}(ip_\mu+\hat{\mathcal{D}}_\mu).$$

"1PI" diagram

- Then

$$= 2g_0^2 \int_0^t d\xi\, \xi^{-D/2} \int_p e^{-2p^2}$$
$$\times \operatorname{tr}\Bigl[\xi^{-1}\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\left(ip_\alpha + \sqrt{\xi}\hat{\mathcal{D}}_\alpha\right)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\beta\gamma}\left(ip_\delta + \sqrt{\xi}\hat{\mathcal{D}}_\delta\right)$$
$$+ \hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\rho\mu}\Bigr].$$

- The second term is

$$2g_0^2 \int_0^t d\xi\, \xi^{-D/2} \int_p e^{-2p^2} \operatorname{tr}\left[\hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\rho\nu}\right]$$
$$= 8g_0^2 \int_0^t d\xi\, \xi^{-D/2+1} \int_p e^{-2p^2} \operatorname{tr}\left[\hat{\mathcal{F}}_{\mu\rho}(x)\hat{\mathcal{F}}_{\rho\nu}(x)\right] + O(t^{3-D/2})$$
$$= \frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{\varepsilon} 2\operatorname{tr}\left[\hat{\mathcal{F}}(x)^2\right]_{\mu\nu} + O(t^{1+\varepsilon}).$$

The third term gives rise to the same contribution.

“1PI” diagram

- The first term (without $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$) is,

$$2g_0^2 \int_0^t d\xi\, \xi^{-D/2-1} \int_p e^{-2p^2}$$
$$\times \operatorname{tr}\left[\left(ip_\alpha + \sqrt{\xi}\hat{\mathcal{D}}_\alpha\right)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\beta\gamma}\left(ip_\delta + \sqrt{\xi}\hat{\mathcal{D}}_\delta\right)\right].$$

- After the expansion wrt $\sqrt{\xi}$ and integrations,

$$\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{1-\varepsilon/2}\frac{1}{16t^2}\delta_{\alpha\delta}\delta_{\beta\gamma}\dim G$$
$$+\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{\varepsilon}\operatorname{tr}\Bigg\{-\delta_{\alpha\delta}\hat{\mathcal{F}}(x)^2_{\beta\gamma} - [\hat{\mathcal{D}}_\alpha,\hat{\mathcal{D}}_\delta]\hat{\mathcal{F}}(x)_{\beta\gamma}$$
$$+\frac{1}{12}\delta_{\beta\gamma}\Big[\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon + \hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon - \hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\delta - \hat{\mathcal{D}}_\delta\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\alpha$$
$$-\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon - \hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon + \hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta + \hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha$$
$$-\delta_{\alpha\delta}\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\varphi\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\varphi + \delta_{\alpha\delta}\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\varepsilon\Big]\Bigg\} + O(t^{1+\varepsilon}).$$

"1PI" diagram

- We then use

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = \hat{\mathcal{F}}_{\mu\nu},$$

to yield

$$\begin{aligned}
&\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{1-\varepsilon/2}\frac{1}{16t^2}\delta_{\alpha\delta}\delta_{\beta\gamma}\dim G \\
&\quad + \frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{\varepsilon}\operatorname{tr}\Big[-\delta_{\alpha\delta}\hat{\mathcal{F}}(x)^2_{\beta\gamma} - \hat{\mathcal{F}}(x)_{\alpha\delta}\hat{\mathcal{F}}(x)_{\beta\gamma} \\
&\qquad\qquad - \frac{1}{6}\delta_{\beta\gamma}\hat{\mathcal{F}}(x)^2_{\alpha\delta} + \frac{1}{24}\delta_{\alpha\delta}\delta_{\beta\gamma}\hat{\mathcal{F}}(x)^2_{\rho\rho}\Big] + O(t^{1+\varepsilon}),
\end{aligned}$$

"1PI" diagram

- Finally, taking the contraction with $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$,

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}$$

$$= \frac{g_0^2}{(4\pi)^2}\dim(G)\frac{3}{8t^2}\delta_{\mu\nu} \leftarrow \text{VEV}$$

$$+ \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right] C_A \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) \leftarrow [\zeta_{11}(t) - 1]$$

$$+ \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right] C_A \delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) \leftarrow \zeta_{12}(t)$$

$$+ O(t),$$

where

$$\varepsilon(t)^{-1} \equiv \frac{1}{\varepsilon} + \ln(8\pi t).$$

Small flow time expansion relevant to EMT

- To the one-loop level, we thus have

$$\mathrm{VEV} = \frac{g_0^2}{(4\pi)^2}\dim(G)\frac{3}{8t^2}\delta_{\mu\nu},$$

$$\zeta_{11}(t) = 1 + \frac{g_0^2}{(4\pi)^2}C_A\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right],$$

$$\zeta_{12}(t) = \frac{g_0^2}{(4\pi)^2}C_A\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right].$$

- Plugging these into

$$c_1(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[2 - \zeta_{11}(t)\right], \qquad c_2(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[-\frac{1}{2}\varepsilon\zeta_{12}(t)\right],$$

and making the renormalization in the $\overline{\mathrm{MS}}$ scheme ($\mu \propto 1/\sqrt{t}$),

$$c_1(t) = \frac{1}{g(\mu)^2}\left\{1 + \frac{g(\mu)^2}{(4\pi)^2}\left[-\beta_0 L(\mu,t) - \frac{7}{3}C_A\right]\right\}, \qquad c_2(t) = \frac{1}{(4\pi)^2}\frac{\beta_0}{8},$$

$$\beta_0 = \frac{11}{3}C_A, \qquad L(\mu,t) \equiv \ln(2\mu^2 t) + \gamma_E.$$

Universal formula for EMT

- For the system containing fermions (with Makino, arXiv:1403.4772),

$$T_{\mu\nu}(x) = \lim_{t\to 0}\Biggl\{ c_1(t)\left[\tilde{\mathcal{O}}_{1,\mu\nu}(t,x) - \frac{1}{4}\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)\right] + c_2(t)\tilde{\mathcal{O}}_{2,\mu\nu}(t,x) + c_3(t)\left[\tilde{\mathcal{O}}_{3,\mu\nu}(t,x) - 2\tilde{\mathcal{O}}_{4,\mu\nu}(t,x)\right] + c_4(t)\tilde{\mathcal{O}}_{4,\mu\nu}(t,x) + c_5(t)\tilde{\mathcal{O}}_{5,\mu\nu}(t,x) - \mathrm{VEV}\Biggr\},$$

where

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
$$\tilde{\mathcal{O}}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x),$$
$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu\overleftrightarrow{D}_\nu + \gamma_\nu\overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

Universal formula for EMT

- and

$$c_{1,2}(t) = \frac{1}{g(\mu)^2} \sum_{\ell=0}^{\infty} k_{1,2}^{(\ell)} \left[\frac{g(\mu)^2}{(4\pi)^2}\right]^{\ell}, \qquad c_{3,4,5}(t) = \sum_{\ell=0}^{\infty} k_{3,4,5}^{(\ell)} \left[\frac{g(\mu)^2}{(4\pi)^2}\right]^{\ell},$$

and to the one-loop order ($T_F = Tn_f$)

$$\begin{aligned}
k_1^{(0)} &= 1, & k_1^{(1)} &= -\beta_0 L(\mu,t) - \frac{7}{3} C_A + \frac{3}{2} T_F, \\
k_2^{(0)} &= 0, & k_2^{(1)} &= \frac{1}{4}\left(\frac{11}{6} C_A + \frac{11}{6} T_F\right), \\
k_3^{(0)} &= \frac{1}{4}, & k_3^{(1)} &= \frac{1}{4}\left(\frac{3}{2} + \ln 432\right) C_F, \\
k_4^{(0)} &= 0, & k_4^{(1)} &= \frac{3}{4} C_F, \\
k_5^{(0)} &= -1, & k_5^{(1)} &= -\left[3L(\mu,t) + \frac{7}{2} + \ln 432\right] C_F.
\end{aligned}$$

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.
- “Universality” holds however only when one removes the regulator.
- Thus, we have to take first the continuum limit $a \to 0$ and then the small flow time limit $t \to 0$.
- Practically, we cannot simply take $a \to 0$ and may take t as small as possible in the fiducial window,

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

  The usefulness with presently-accessible lattice parameters is not obvious a priori. . .
- Quite recently, a two-loop computation of $c_i(t)$!!!
  (Harlander–Kluth–Lange, arXiv:1808.09837)

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.
- Here, we presented a construction of a universal formula of the EMT, which may be used in lattice Monte Carlo simulations.
- Possible obstacle would be

$$a \ll \sqrt{8t}.$$

- Nevertheless, bulk thermodynamics (one-point function) shows encouraging results; the method appears promising even practically (see below).
- Further applications to other Noether currents, such as vector/chiral currents and the supercurrent have been considered. . . (Makino–Kasai–Endo–Hieda–Miura–Morikawa–H.S.)