Lectures on the gradient flow
グラディエントフローの基礎とその応用

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Contents

Chapter 1

Quantum theory of the Yang–Mills theory

1.1 Notations

Spacetime signature is $(+,+,+,+)$ (euclidean) and gamma matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{1.1}$$

are all hermitian.

We normalize the gauge group generator as

$$\mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}. \tag{1.2}$$

The structure constant are defined by

$$[T^a, T^b] = f^{abc}T^c, \tag{1.3}$$

and quadratic Casimirs are

$$f^{acd}f^{bcd} = C_A\delta^{ab}, \qquad T^aT^a = -C_F\mathbb{1}. \tag{1.4}$$

We also use the following abbreviation for the momentum integral:

$$\int_p \equiv \int \frac{d^Dp}{(2\pi)^D}, \tag{1.5}$$

where we set the spacetime dimension D having use of the dimensional regularization, in which $D = 4-2\epsilon$, in mind.

Dimensional regularization

A basic formula in the dimensional regularization is

$$\int_\ell \frac{1}{(\ell^2+M^2)^\alpha} = \frac{\Gamma(\alpha - D/2)}{(4\pi)^{D/2}\Gamma(\alpha)} M^{2(D/2-\alpha)}. \tag{1.6}$$

In particular (setting $D = 4-2\epsilon$),

$$\int_\ell \frac{1}{(\ell^2+M^2)^2} = \frac{1}{(4\pi)^2}\Gamma(2-D/2)\left(\frac{M^2}{4\pi}\right)^{(D/2-2)} = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma_E - \ln\left(\frac{M^2}{4\pi}\right)\right], \tag{1.7}$$

and

$$\int_\ell \frac{1}{\ell^2+M^2} = \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}} M^{D-2} \overset{M\to 0}{\to} 0, \tag{1.8}$$

and

$$\int_\ell 1 = \frac{\Gamma(-D/2)}{(4\pi)^{D/2}} M^D \stackrel{M\to 0}{\to} 0. \tag{1.9}$$

Note that the last quantity is nothing but $\delta(0)$.

1.2 Functional integral and the Faddeev–Popov ghost

1.2.1 Quantization of the scalar field

For a real scalar field $\varphi(x)$, the functional integral

$$\mathcal{Z} \equiv \int \mathcal{D}\varphi\, e^{-S}, \qquad S = \int d^D x \left[\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + V(\varphi)\right], \tag{1.10}$$

defines the quantum theory. The perturbation theory is simply developed by expanding the potential around a (constant) stationary point φ_0 such that $V'(\varphi_0) = 0$ as

$$\mathcal{Z} = \int \mathcal{D}\varphi\, e^{-S_0 - S_{\text{int}}}, \tag{1.11}$$

where (under $\varphi(x) \to \varphi(x) + \varphi_0$),

$$S_0 = \int d^D x \left(\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{1}{2}m_0^2\varphi^2\right), \qquad m_0^2 \equiv V''(\varphi_0), \tag{1.12}$$

$$S_{\text{int}} = \int d^D x \sum_{n=3}^{\infty} \frac{1}{n!} V^{(n)}(\varphi_0)\varphi^n. \tag{1.13}$$

Now, take

$$\int \mathcal{D}\varphi\, e^{-S_0}\varphi(y) \tag{1.14}$$

and consider the infinitesimal shift of the integration variable, $\varphi \to \varphi + \delta\varphi$. The integral itself does not change under this and, if the integration measure is invariant under the shift, we have the identity (a Schwinger–Dyson equation)

$$\left\langle (-1)\int d^D x\, \delta\varphi(x)\left(-\partial_\mu\partial_\mu + m_0^2\right)\varphi(x)\varphi(y) + \delta\varphi(y)\right\rangle_0 = 0, \tag{1.15}$$

where

$$\langle \mathcal{O} \rangle_0 \equiv \frac{\int \mathcal{D}\varphi\, e^{-S_0}\mathcal{O}}{\int \mathcal{D}\varphi\, e^{-S_0}}. \tag{1.16}$$

Taking the derivative w.r.t. $\delta\varphi(x)$, that implies

$$(-1)\left(-\partial_\mu\partial_\mu + m_0^2\right)\langle\varphi(x)\varphi(y)\rangle_0 + \delta(x-y) = 0. \tag{1.17}$$

This gives the *free propagator*:

$$\langle\varphi(x)\varphi(y)\rangle_0 = \frac{1}{-\partial_\mu\partial_\mu + m_0^2}\delta(x-y) = \int_p \frac{e^{ip(x-y)}}{p^2 + m_0^2}. \tag{1.18}$$

Presumably, this is the quickest way to obtain the free propagator without mistake. Then the correlation functions of φ can be perturbatively computed by expanding $e^{-S_{\text{int}}}$ w.r.t. φ and contracting φ's by the free propagator.

We will later use the notion of the effective action, the generating functional of the one particle irreducible (1PI) correlation functions. To define this, we first introduce the generating functional of the connected correlation functions $W[J]$:

$$e^{-W[J]} \equiv \int \mathcal{D}\varphi\, e^{-S+J\cdot\varphi}, \qquad J\cdot\varphi \equiv \int d^Dx\, J(x)\varphi(x). \tag{1.19}$$

The expectation value of φ *in the presence of* J is given by

$$\phi(x) \equiv \langle\varphi(x)\rangle = -\frac{\delta}{\delta J(x)} W[J]. \tag{1.20}$$

The effective action $\Gamma[\phi]$ is defined by the Legendre transformation of $W[J]$, that is

$$\Gamma[\phi] \equiv W[J] + J\cdot\phi. \tag{1.21}$$

As usual for the Legendre transformation, we have a relation dual to the above:

$$J(x) = \frac{\delta}{\delta\phi(x)}\Gamma[\phi]. \tag{1.22}$$

1.2.2 Quantization of the Yang–Mills field

The action of the Yang–Mills theory is

$$S = \frac{1}{4g_0^2}\int d^Dx\, F^a_{\mu\nu}F^a_{\mu\nu}. \tag{1.23}$$

where

$$F_{\mu\nu}(x) \equiv \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + f^{abc}A^b_\mu(x)A^c_\nu(x), \tag{1.24}$$

is called the *field strength*. This is a gauge theory, i.e., the action is invariant under the following *gauge transformation*,

$$\delta A^a_\mu(x) = \partial_\mu\omega^a(x) + f^{abc}A^b_\mu(x)\omega^c(x) \equiv D_\mu\omega^a(x). \tag{1.25}$$

This describes, for example, the system of the gluon, and thus is very important.

The gauge invariance, however, prevents a simple perturbation theory. The would-be “free propagator”

$$\left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle_0 \stackrel{?}{=} \frac{\int\mathcal{D}A\, e^{-S_0}A^a_\mu(x)A^b_\nu(y)}{\int\mathcal{D}A\, e^{-S_0}}, \tag{1.26}$$

where

$$S_0 \equiv \frac{1}{4g_0^2}\int d^Dx\, \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2. \tag{1.27}$$

But if this free propagator is well-defined, we have

$$\partial_\mu\omega^a(x)\underbrace{\left\langle A^b_\nu(y)\right\rangle_0}_{=0} + \partial_\nu\omega^b(y)\underbrace{\left\langle A^a_\mu(x)\right\rangle_0}_{=0} + \partial_\mu\omega^a(x)\partial_\mu\omega^b(y) = 0, \tag{1.28}$$

that is, $\partial_\mu\omega^a(x)\partial_\mu\omega^a(x) = 0$, a contradiction. Therefore, that expression must be ill-defined. To develop the perturbation theory, we have to break the gauge invariance by adding the (Lorentz) gauge fixing term,

$$S_{\mathrm{gf}} = \frac{\lambda_0}{2g_0^2}\int d^Dx\, \left(\partial_\mu A^a_\mu\right)^2 \tag{1.29}$$

and the ghost–anti-ghost term $S_{c\bar{c}}$ associated to this (the Faddeev–Popov prescription).

It is known that required terms can be summarized as

$$S_{\rm gf} + S_{c\bar{c}} = \delta \frac{1}{g_0^2} \int d^D x\, \bar{c}^a \left(\partial_\mu A_\mu^a - \frac{1}{2\lambda_0} B^a \right) \tag{1.30}$$

$$= \frac{1}{g_0^2} \int d^D x \left(B^a \partial_\mu A_\mu^a - \frac{1}{2\lambda_0} B^a B^a + \frac{1}{2\lambda_0} \bar{c}^a \partial_\mu D_\mu c^a \right), \tag{1.31}$$

where δ denotes the nilpotent BRS transformation,

$$\delta A_\mu^a(x) = D_\mu c^a(x), \qquad \delta c^a(x) = -\frac{1}{2} f^{abc} c^b(x) c^c(x), \tag{1.32}$$

$$\delta \bar{c}^a(x) = B^a(x), \qquad \delta B^a(x) = 0. \tag{1.33}$$

If we eliminates the Nakanishi-Lautrup auxiliary field by the equation of motion,

$$B^a = \lambda_0 \partial_\mu A_\mu^a, \tag{1.34}$$

we have the above gauge fixing term. The actions are BRS invariant:

$$\delta S = 0, \qquad \delta(S_{\rm gf} + S_{c\bar{c}}) = 0. \tag{1.35}$$

This shows that any correlation functions of gauge invariant operators are independent of the gauge fixing parameter λ_0:

$$\begin{aligned} \frac{\partial}{\partial \lambda_0} \langle \mathcal{O} \rangle &= -\frac{1}{2\lambda_0^2 g_0^2} \int d^D x\, \langle \delta \left[\bar{c}^a(x) B^a(x) \right] \mathcal{O} \rangle \\ &= -\frac{1}{2\lambda_0^2 g_0^2} \int d^D x\, \langle \bar{c}^a(x) B^a(x) \delta \mathcal{O} \rangle \qquad \because \langle \delta(\text{anything}) \rangle = 0 \\ &= 0 \qquad \because \mathcal{O} \text{ is gauge (i.e., BRS) invariant.} \end{aligned} \tag{1.36}$$

This shows that the gauge fixing term has no effect on physical quantities.

1.3 Feynman rules and the 1-loop calculation

1.3.1 Free propagators and the interaction vertices

The action is

$$\begin{aligned} S + S_{\rm gf} &= \frac{1}{4g_0^2} \int d^D x\, F_{\mu\nu}^a F_{\mu\nu}^a + \frac{\lambda_0}{2g_0^2} \int d^D x\, (\partial_\mu A_\mu^a)^2 \\ &\equiv S_{\rm free} + S_{\rm int}, \end{aligned} \tag{1.37}$$

where

$$S_{\rm free} = \frac{1}{g_0^2} \int d^D x\, \frac{1}{2} A_\mu^a \delta^{ab} (-1) [\delta_{\mu\nu} \partial_\rho \partial_\rho + (\lambda_0 - 1) \partial_\mu \partial_\nu] A_\nu^b, \tag{1.38}$$

and

$$S_{\rm int} = \frac{1}{g_0^2} \int d^D x \left(f^{abc} \partial_\mu A_\nu^a A_\mu^b A_\nu^c + \frac{1}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e \right). \tag{1.39}$$

The free propagator: Taking the functional integral

$$\int \mathcal{D} A_\mu\, e^{-S_{\rm free}} A_\nu^b(y), \tag{1.40}$$

and consider the variation of integration variable, $A^a_\mu \to A^a_\mu + \delta A^a_\mu$. This gives the Schwinger–Dyson equation,

$$-\frac{1}{g_0^2}\delta^{ca}(-1)\left[\delta_{\rho\mu}\partial_\sigma\partial_\sigma + (\lambda_0 - 1)\partial_\rho\partial_\mu\right]\langle A^a_\mu(x)A^b_\nu(y)\rangle_0 + \delta^{cb}\delta_{\rho\nu}\delta(x-y) = 0. \tag{1.41}$$

The solution is

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \frac{1}{\lambda_0}p_\mu p_\mu\right]. \tag{1.42}$$

(**Exercise:** Verify this.)

The ghost action is

$$\begin{aligned} S_{c\bar{c}} &= -\frac{1}{g_0^2}\int d^Dx\,\bar{c}^a\partial_\mu D_\mu c^a &(1.43)\\ &= -\frac{1}{g_0^2}\int d^Dx\,\bar{c}^a\partial_\mu\partial_\mu c^a + \frac{1}{g_0^2}\int d^Dx\,f^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c, &(1.44)\end{aligned}$$

and the free propagator is given by

$$\left\langle c^a(x)\bar{c}^b(y)\right\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2} \tag{1.45}$$

1.3.2 Computation of the divergent part in the 1-loop level

Throughout these lectures, we adopt the Feynman gauge $\lambda_0 = 1$ for computational simplicity.

Ghost two-point function

Let us start with the simplest one: the ghost two-point function. The expansion of the exponential factor gives rise to

$$\left\langle c^a(x)\bar{c}^b(y)\frac{1}{2!}\frac{1}{g_0^4}\int d^Dz\int d^Dw\,f^{cde}f^{fgh}\partial_\mu\bar{c}^c(z)A^d_\mu(z)c^e(z)\partial_\nu\bar{c}^f(w)A^g_\nu(w)c^h(w)\right\rangle_0. \tag{1.46}$$

After the contraction by the free propagators,

$$g_0^4 f^{ade}f^{edb}\int d^Dz\int d^Dw\int_k\frac{e^{ik(x-z)}}{k^2}\int_{k'}\frac{e^{ik'(w-y)}}{(k')^2}\int_p\frac{e^{ip(z-w)}}{p^2}\int_q\frac{e^{iq(z-w)}}{q^2}(-ik)\cdot(-iq). \tag{1.47}$$

The integration over z and w imposes

$$k' = -k, \qquad q \equiv \ell, \qquad p = -\ell + k. \tag{1.48}$$

Then,

$$g_0^4 C_A\delta^{ab}\int_k\frac{e^{ik(x-y)}}{(k^2)^2}\int_\ell\frac{1}{\ell^2(\ell-k)^2}k\cdot\ell. \tag{1.49}$$

To this, we apply Feynman's parameter formula

$$\begin{aligned}\frac{1}{\ell^2(\ell-k)^2} &= \int_0^1 dx\,\frac{1}{[\ell^2(1-x)+(\ell-k)^2x]^2}\\ &= \int_0^1 dx\,\frac{1}{[(\ell')^2+k^2x(1-x)]^2}, \qquad \ell = \ell' + kx. &(1.50)\end{aligned}$$

Then (we set $\ell' \to \ell$),

$$g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_0^1 dx \int_\ell \frac{1}{[\ell^2 + k^2 x(1-x)]^2} k \cdot (\ell + kx). \tag{1.51}$$

Since

$$\frac{1}{[\ell^2 + k^2 x(1-x)]^2} = \frac{1}{(\ell^2)^2} + \frac{1}{(\ell^2)^3}(-2)k^2 x(1-x) + O(\ell^{-8}), \tag{1.52}$$

the divergent part is given by

$$g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_0^1 dx \int_\ell \frac{1}{(\ell^2)^2} k^2 x. \tag{1.53}$$

In the dimensional regularization,

$$\int_\ell \frac{1}{(\ell^2)^2} = \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \sim \frac{1}{(4\pi)^2} \frac{1}{\epsilon}. \tag{1.54}$$

That is, we find

$$\left\langle c^a(x) \bar{c}^b(y) \right\rangle_{\text{1-loop, divergent}} = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \frac{g_0^2}{(4\pi)^2} \frac{1}{2} C_A \frac{1}{\epsilon} p^2. \tag{1.55}$$

Gauge field two-point function

Next we consider the two-point function of the gauge field; this is somewhat tough. For this, it is convenient to represent the 3-point vertex in the momentum space:

$$[(A_\mu^a, k), (A_\nu^b, p), (A_\rho^c, q)] = -\frac{i}{g_0^2} f^{abc} [\delta_{\mu\nu}(k-p)_\rho + \delta_{\nu\rho}(p-q)_\mu + \delta_{\rho\mu}(q-k)_\nu]. \tag{1.56}$$

In the first diagram, after making the contraction,

$$\begin{aligned}(-i)^2 g_0^4 f^{acd} f^{bdc} \int d^D z \int d^D w \int_k \frac{e^{ik(x-z)}}{k^2} \int_{k'} \frac{e^{ik'(y-w)}}{(k')^2} \int_p \frac{e^{ip(w-z)}}{p^2} \int_q \frac{e^{iq(w-z)}}{q^2} \\ \times \frac{1}{2} \left[\delta_{\mu\rho}(k-p)_\sigma + \delta_{\rho\sigma}(p-q)_\mu + \delta_{\sigma\mu}(q-k)_\rho\right] \left[\delta_{\nu\sigma}(k'+q)_\rho + \delta_{\sigma\rho}(-q+p)_\nu + \delta_{\rho\nu}(-p-k')_\sigma\right].\end{aligned} \tag{1.57}$$

After the integration over z and w,

$$k' = -k, \qquad q \equiv -\ell, \qquad p = \ell - k. \tag{1.58}$$

Then,

$$\begin{aligned}\frac{1}{2} g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_\ell \frac{1}{\ell^2 (\ell-k)^2} \\ \times [\delta_{\mu\rho}(-\ell+2k)_\sigma + \delta_{\rho\sigma}(2\ell-k)_\mu + \delta_{\sigma\mu}(-\ell-k)_\rho][\delta_{\nu\sigma}(-\ell-k)_\rho + \delta_{\sigma\rho}(2\ell-k)_\nu + \delta_{\rho\nu}(-\ell+2k)_\sigma].\end{aligned} \tag{1.59}$$

Then, after some calculation by using the Feynman's parameter formula (we set $\ell' \to \ell$),

$$\begin{aligned}\frac{1}{2} g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_0^1 dx \int_\ell \frac{1}{[\ell^2 + k^2 x(1-x)]^2} \\ \times \left\{2\delta_{\mu\nu}\ell^2 + (4D-6)\ell_\mu \ell_\nu + (2x^2 - 2x + 5)\delta_{\mu\nu}k^2 + \left[D(2x-1)^2 - 6(x^2 - x + 1)\right] k_\mu k_\nu\right\}.\end{aligned} \tag{1.60}$$

We can set $\ell_\mu\ell_\nu \to \frac{1}{D}\delta_{\mu\nu}\ell^2$ and in the divergent part, $D \to 4$. Then the divergent part is given by

$$g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_\ell \frac{1}{(\ell^2)^2} \left(\frac{19}{12}\delta_{\mu\nu}k^2 - \frac{11}{6}k_\mu k_\nu \right). \tag{1.61}$$

The divergent part of the first diagram is thus

$$g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \frac{g_0^2}{(4\pi)^2} \frac{1}{\epsilon} C_A \left(\frac{19}{12}\delta_{\mu\nu}p^2 - \frac{11}{6}p_\mu p_\nu \right). \tag{1.62}$$

The second diagram including the ghost loop reads

$$\left\langle A_\mu^a(x) A_\nu^b(y) \frac{1}{2!}\frac{1}{g_0^4} \int d^D z \int d^D w\, f^{cde} f^{fgh} \partial_\rho \bar{c}^c(z) A_\rho^d(z) c^e(z) \partial_\sigma \bar{c}^f(w) A_\sigma^g(w) c^h(w) \right\rangle_0 . \tag{1.63}$$

After the contraction,

$$(-1) g_0^4 f^{cae} f^{ebc} \int d^D z \int d^D w \int_k \frac{e^{ik(x-z)}}{k^2} \int_{k'} \frac{e^{ik'(y-w)}}{(k')^2} \int_p \frac{e^{ip(w-z)}}{p^2} \int_q \frac{e^{iq(z-w)}}{q^2} (-ip)_\mu (-iq)_\nu . \tag{1.64}$$

After the integration over z and w,

$$k' = -k, \qquad p \equiv \ell, \qquad q = \ell + k. \tag{1.65}$$

Repeating a similar procedure as above,

$$-g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_0^1 dx \int_\ell \frac{1}{[\ell^2 + k^2 x(1-x)]^2} [\underbrace{\ell_\mu \ell_\nu}_{\frac{1}{D}\delta_{\mu\nu}\ell^2} - x(1-x) k_\mu k_\nu]. \tag{1.66}$$

Then the divergent part of the second diagram is given by

$$\begin{aligned} & g_0^4 C_A \delta^{ab} \int_k \frac{e^{ik(x-y)}}{(k^2)^2} \int_\ell \frac{1}{(\ell^2)^2} \left(\frac{1}{12}\delta_{\mu\nu}k^2 + \frac{1}{6}k_\mu k_\nu \right) \\ &= g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \frac{g_0^2}{(4\pi)^2} \frac{1}{\epsilon} C_A \left(\frac{1}{12}\delta_{\mu\nu}p^2 + \frac{1}{6}p_\mu p_\nu \right). \end{aligned} \tag{1.67}$$

Finally, the third diagram vanishes under the dimensional regularization with which

$$\int_\ell \frac{1}{\ell^2} = 0, \tag{1.68}$$

and thus the sum of divergent parts of the three diagrams is

$$\left\langle A_\mu^a(x) A_\nu^b(y) \right\rangle_{\text{1-loop, divergent}} = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \frac{g_0^2}{(4\pi)^2} \frac{1}{\epsilon} \frac{5}{3} C_A \left(\delta_{\mu\nu} p^2 - p_\mu p_\nu \right). \tag{1.69}$$

ghost-anti-ghost-gauge three-point function

In the tree level,

$$\begin{aligned} & \left\langle c^a(x) A_\mu^b(y) \bar{c}^c(z) \right\rangle_{\text{tree}} \\ &= g_0^4 f^{abc} \int_{p,q} e^{ip(x-z)} e^{iq(y-z)} \frac{1}{p^2} \frac{1}{(p+q)^2} \frac{1}{(q^2)^2} \left[(\delta_{\mu\nu} q^2 - q_\mu q_\nu) + \frac{1}{\lambda_0} q_\mu q_\nu \right] i p_\nu . \end{aligned} \tag{1.70}$$

In the one-loop level, the first 1PI diagram containing AAA vertex, yields after some calculation,

$$\begin{aligned}&\left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{one-loop, divergent}}\\&=g_0^4 f^{abc}\int_{k,q} e^{ip(x-z)}e^{iq(y-z)}\frac{1}{p^2}\frac{1}{(p+q)^2}\frac{1}{q^2}\left[\frac{g_0^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{3}{8}C_A\right]ip_\mu.\end{aligned}\tag{1.71}$$

To obtain this, we have to use the identity

$$f^{aXb}f^{bYc}f^{cZa}=-\frac{1}{2}f^{XYZ}C_A.\tag{1.72}$$

The second 1PI diagram containing three $\bar{c}Ac$ vertices yields

$$\begin{aligned}&\left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{one-loop, divergent}}\\&=g_0^4 f^{abc}\int_{k,q} e^{ip(x-z)}e^{iq(y-z)}\frac{1}{p^2}\frac{1}{(p+q)^2}\frac{1}{q^2}\left[\frac{g_0^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{1}{8}C_A\right]ip_\mu.\end{aligned}\tag{1.73}$$

We also have three one-particle *reducible* diagrams. From the results for two-point functions, we have

$$\begin{aligned}&\left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{1PR, one-loop, divergent}}\\&=g_0^4 f^{abc}\int_{p,q} e^{ip(x-z)}e^{iq(y-z)}\frac{1}{p^2}\frac{1}{(p+q)^2}\frac{1}{(q^2)^2}\frac{g_0^2}{(4\pi)^2}\frac{1}{\epsilon}C_A\left[\left(\frac{5}{3}+2\cdot\frac{1}{2}\right)\delta_{\mu\nu}q^2-\frac{5}{3}q_\mu q_\nu\right]ip_\nu.\end{aligned}\tag{1.74}$$

In total,

$$\begin{aligned}&\left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{one-loop, divergent}}\\&=g_0^4 f^{abc}\int_{p,q} e^{ip(x-z)}e^{iq(y-z)}\frac{1}{p^2}\frac{1}{(p+q)^2}\frac{1}{(q^2)^2}\frac{g_0^2}{(4\pi)^2}\frac{1}{\epsilon}C_A\left(\frac{19}{6}\delta_{\mu\nu}q^2-\frac{5}{3}q_\mu q_\nu\right)ip_\nu,\end{aligned}\tag{1.75}$$

in the Feynman gauge $\lambda_0=1$.

1.4 Renormalization

1.4.1 Renormalization constants

Later, we will prove that all UV divergences appearing in correlation functions of elementary fields can be absorbed into the following 3 constants, Z, Z_3, and $\tilde{Z}_3$,

$$\begin{aligned}g_0^2&=\mu^{2\varepsilon}g^2Z,\\\lambda_0&=\lambda Z_3^{-1},\\A^a_\mu&=Z^{1/2}Z_3^{1/2}(A_R)^a_\mu,\\c^a&=\tilde{Z}_3Z^{1/2}Z_3^{1/2}c^a_R,\\\bar{c}^a&=Z^{1/2}Z_3^{-1/2}\bar{c}^a_R,\end{aligned}\tag{1.76}$$

order by order in perturbation theory, as

$$Z=1+Z^{(1)}+Z^{(2)}+\cdots,\tag{1.77}$$

etc.

1.4.2 One-loop determination of renormalization constants

To the one-loop level,

$$\begin{aligned}
&\left\langle (A_R)^a_\mu(x)(A_R)^b_\nu(y)\right\rangle \\
&= Z^{-1}Z_3^{-1}\left[\left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle_{\text{tree}} + \left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle_{\text{one-loop}}\right] \\
&= \mu^{2\epsilon}g^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[Z_3^{-1}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \frac{1}{\lambda}p_\mu p_\nu\right] + \left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle_{\text{one-loop}}.
\end{aligned} \tag{1.78}$$

Thus, by choosing

$$Z_3^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{5}{3}C_A, \tag{1.79}$$

the one-loop UV divergence in $\langle A^a_\mu(x)A^b_\nu(y)\rangle$ is removed. Note that in the one-loop contribution we can regard $g_0 = g$.

Similarly,

$$\begin{aligned}
&\left\langle c^a_R(x)\bar{c}^b_R(y)\right\rangle \\
&= \tilde{Z}_3^{-1}Z^{-1}\left[\left\langle c^a(x)\bar{c}^b(y)\right\rangle_{\text{tree}} + \left\langle c^a(x)\bar{c}^b(y)\right\rangle_{\text{one-loop}}\right] \\
&= \mu^{2\epsilon}g^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}\tilde{Z}_3^{-1} + \left\langle c^a(x)\bar{c}^b(y)\right\rangle_{\text{one-loop}}.
\end{aligned} \tag{1.80}$$

Thus,

$$\tilde{Z}_3^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{1}{2}C_A, \tag{1.81}$$

removes the divergence.

The last constant Z can be deduced from

$$\begin{aligned}
&\left\langle c^a_R(x)(A_R)^b_\mu(y)\bar{c}^c_R(z)\right\rangle \\
&= \tilde{Z}_3^{-1}Z^{-3/2}Z_3^{-1/2}\left[\left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{tree}} + \left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{one-loop}}\right] \\
&= \mu^{4\epsilon}g^4 f^{abc}\int_{p,q} e^{ip(x-z)}e^{iq(y-z)} \\
&\quad\times \frac{1}{p^2}\frac{1}{(p+q)^2}\frac{1}{(q^2)^2}\left[\tilde{Z}_3^{-1}Z^{1/2}Z_3^{-1/2}(\delta_{\mu\nu}q^2 - q_\mu q_\nu) + \tilde{Z}_3^{-1}Z^{1/2}Z_3^{1/2}\frac{1}{\lambda}q_\mu q_\nu\right] ip_\nu \\
&\qquad + \left\langle c^a(x)A^b_\mu(y)\bar{c}^c(z)\right\rangle_{\text{one-loop}}.
\end{aligned} \tag{1.82}$$

The cancellation of the divergences requires

$$\tilde{Z}_3^{(1)} - \frac{1}{2}Z^{(1)} + \frac{1}{2}Z_3^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{19}{6}C_A, \tag{1.83}$$

$$-\left(\tilde{Z}_3^{(1)} - \frac{1}{2}Z^{(1)} + \frac{1}{2}Z_3^{(1)}\right) + \tilde{Z}_3^{(1)} - \frac{1}{2}Z^{(1)} - \frac{1}{2}Z_3^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{5}{3}\right)C_A. \tag{1.84}$$

These yield again $Z_3^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\frac{5}{3}C_A$ and

$$Z^{(1)} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{11}{3}\right)C_A. \tag{1.85}$$

The minus sign in this expression shows the famous asymptotic freedom.

Thus we have examined the renormalizability of the Yang–Mills theory and determined the renormalization constants to the one-loop level.

Let us summarize the one-loop renormalization in the Feynman gauge:

$$\begin{aligned}
g_0^2 &= \mu^{2\varepsilon} g^2 \left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{11}{3}\right) C_A\right], \\
\lambda_0 &= \lambda \left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{5}{3}\right) C_A\right], \\
A_\mu^a &= \left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}(-1) C_A\right] (A_R)_\mu^a, \\
c^a &= \left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{1}{2}\right) C_A\right] c_R^a, \\
\bar{c}^a &= \left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{8}{3}\right) C_A\right] \bar{c}_R^a.
\end{aligned} \tag{1.86}$$

The renormalization always has the ambiguity that one may subtract any finite quantity in addition to the above; this freedom in the renormalization prescription is called the renormalization scheme. The above prescription is termed the minimal subtraction (MS) scheme. In phenomenology, it is quite common to adopt the modified minimal subtraction ($\overline{\text{MS}}$) scheme instead that is simply the MS scheme with the change of the renormalization scale μ:

$$\mu^2 \to 4\pi e^{-\gamma_E} \mu^2. \tag{1.87}$$

Chapter 2

Gradient flow

2.1 Yang–Mills gradient flow

The Yang–Mills gradient flow is the evolution of the gauge field $A^a_\mu(x)$ along a fictitious time $t \in [0, \infty)$, according to

$$\partial_t B^a_\mu(t,x) = -g_0^2 \frac{\delta S}{\delta B^a_\mu(t,x)} = D_\nu G^a_{\nu\mu}(t,x) = \Delta B^a_\mu(t,x) + \cdots , \tag{2.1}$$

where

$$G^a_{\mu\nu}(t,x) = \partial_\mu B^a_\nu(t,x) - \partial_\nu B^a_\mu(t,x) + f^{abc} B^b_\mu(t,x) B^c_\nu(t,x), \tag{2.2}$$

$$D_\nu G^a_{\nu\mu}(t,x) = \partial_\nu G^a_{\nu\mu}(t,x) + f^{abc} B^b_\nu(t,x) G^c_{\nu\mu}(t,x). \tag{2.3}$$

The initial value is thus the conventional gauge field

$$B^a_\mu(t=0,x) = A^a_\mu(x). \tag{2.4}$$

The RHS of the flow equation is the Yang–Mills equation of motion, the gradient in functional space if S is regarded as a potential function. So the name of the gradient flow. The flow equation is a sort of diffusion equation in which the diffusion length is

$$x \sim \sqrt{8t}. \tag{2.5}$$

Thus the flow makes field configurations smooth.

2.2 Perturbative expansion of the gradient flow

The flow equation reads

$$\partial_t B^a_\mu(t,x) = D_\nu G^a_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B^a_\nu(t,x), \qquad B^a_\mu(t=0,x) = A^a_\mu(x), \tag{2.6}$$

where the term with α_0 is introduced to suppress gauge modes.

2.2.1 Justification of the "gauge fixing term"

Under the infinitesimal gauge transformation,x

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu \omega(t,x), \tag{2.7}$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \tag{2.8}$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x). \tag{2.9}$$

Choosing $\omega(t,x)$ as

$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = -\delta\alpha_0 \partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0, \tag{2.10}$$

α_0 can be changed accordingly

$$\alpha_0 \to \alpha_0 + \delta\alpha_0. \tag{2.11}$$

Thus, a gauge invariant quantity (in usual 4D sense is independent of α_0, as far as it does not contain the flow time derivative ∂_t.

The flow equation can be formally solved as

$$B^a_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A^a_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R^a_\nu(s,y) \right], \tag{2.12}$$

by using the heat kernel,

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right]. \tag{2.13}$$

R represents the non-linear terms in the flow equation,

$$R^a_\mu = f^{abc} \left[2 B^b_\nu \partial_\nu B^c_\mu - B^b_\nu \partial_\mu B^c_\nu + (\alpha_0 - 1) B^b_\mu \partial_\nu B^c_\nu \right] + f^{abc} f^{cde} B^b_\nu B^d_\nu B^e_\mu. \tag{2.14}$$

The solution is represented diagrammatically as (double lines: K, crosses: A_μ, white circles: R),

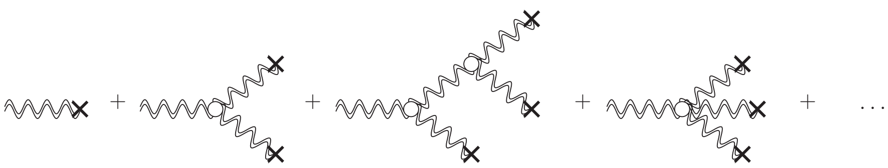
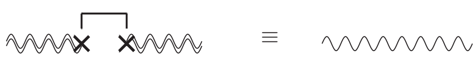


Fig. 2.1

So far, we have discussed the classical theory. Quantum correlation functions of the flowed gauge field are obtained by the functional integral over the initial value $A_\mu(x)$:

$$\langle B_{\mu_1}(t_1,x_1) \cdots B_{\mu_n}(t_n,x_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu\, B_{\mu_1}(t_1,x_1) \cdots B_{\mu_n}(t_n,x_n)\, e^{-S - S_{\mathrm{gf}} - S_{c\bar{c}}}. \tag{2.15}$$

For example, the contraction of two A_μ's



yields the free propagator of the flowed field, as

$$
\begin{aligned}
\left\langle B_\mu^a(t,x) B_\nu^b(s,y) \right\rangle_0 = &\int d^D z \int_p \frac{e^{ip(x-z)}}{p^2} \left[(\delta_{\mu\rho} p^2 - p_\mu p_\rho) e^{-tp^2} + p_\mu p_\rho e^{-\alpha_0 t p^2} \right] \\
&\times \int d^D w \int_q \frac{e^{iq(y-w)}}{q^2} \left[(\delta_{\nu\sigma} q^2 - q_\nu q_\sigma) e^{-sq^2} + q_\nu q_\sigma e^{-\alpha_0 s q^2} \right] \\
&\times g_0^2 \delta^{ab} \int_k \frac{e^{ik(z-w)}}{(k^2)^2} \left[(\delta_{\rho\sigma} k^2 - k_\rho k_\sigma) + \frac{1}{\lambda_0} k_\rho k_\sigma \right] . \qquad (2.16)
\end{aligned}
$$

After the integration over z and w, $p = -q = k$ and we have

$$
\left\langle B_\mu^a(t,x) B_\nu^b(s,y) \right\rangle_0 = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0 (t+s) p^2} \right] . \qquad (2.17)
$$

Note that this free propagator contains Gaussian damping factors.

Similarly, for (black circle: Yang–Mills vertex)

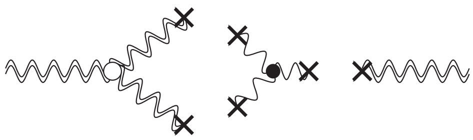


Fig. 2.2

we have the loop flow-line Feynman diagram

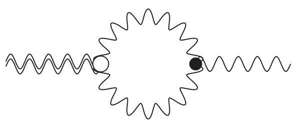


Fig. 2.3

Recall that the flowed gauge field is represented as

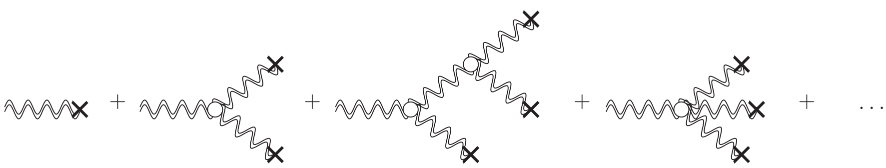


Fig. 2.4

This expansion can be regarded as the loop expansion, because

$$
B_\mu^a(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu^a(y) + \int_0^t ds\ \underbrace{g_0^2 K_{t-s}(x-y)_{\mu\nu}}_{\text{internal line}} \underbrace{\frac{1}{g_0^2} R_\nu^a(s,y)}_{\text{flow vertex}} \right] , \qquad (2.18)
$$

and

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right]. \tag{2.19}$$

2.2.2 Two-point function of flowed gauge field

Now, let us study the two-point function of the flowed gauge field

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle \tag{2.20}$$

to the one-loop. We assume that t, $s > 0$.

In the tree level, as we have observed,

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_{\text{tree}} = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right]. \tag{2.21}$$

One-loop diagrams that consist only of Yang–Mills vertices are

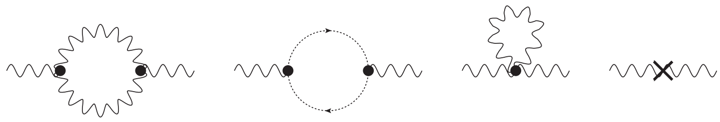


Fig. 2.5

where the last counter term arises from the parameter renormalization,

$$g_0^2 = \mu^{2\varepsilon}g^2\left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{11}{3}\right)C_A\right],$$
$$\lambda_0 = \lambda\left[1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(-\frac{5}{3}\right)C_A\right], \tag{2.22}$$

are obtained by our previous calculation in the Yang–Mills theory simply putting the Gaussian factors to the external lines. In what follows, we set (the "Feynman gauge")

$$\lambda_0 = \alpha_0 = 1. \tag{2.23}$$

We have (we used the fact that α_0 does not receive the renormalization)

$$\left\langle B_\mu^a(t,x)B_\nu^b(t,y)\right\rangle_{\text{1-loop, divergent+parameter ren.}}$$
$$= \mu^{2\epsilon}g^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}e^{-(t+s)p^2}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}(-2)C_A\delta_{\mu\nu}p^2. \tag{2.24}$$

If we also included the *wave function renormalization*, this remaining divergence would be absent.

In the present flowed system, however, we also have diagrams containing the white circles (flow vertex)

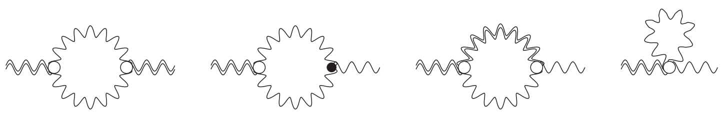


Fig. 2.6

Let us start with the last diagram. It reads

$$\int d^D z \int_0^t du \int_p e^{ip(x-z)} e^{-(t-u)p^2} f^{acd} f^{def} \left\langle B_\rho^c(u,z) B_\rho^e(u,z) B_\mu^f(u,z) B_\nu^b(s,y) \right\rangle_0 . \tag{2.25}$$

After the contraction,

$$g_0^4 \int d^D z \int_0^t du \int_p e^{ip(x-z)} e^{-(t-u)p^2} \int_q \frac{e^{iq(y-z)}}{q^2} e^{-(s+u)q^2} \int_\ell \frac{1}{\ell^2} e^{-2u\ell^2} (f^{acd} f^{dcb} D + f^{acd} f^{dbc}) \delta_{\mu\nu}. \tag{2.26}$$

After the integration over z, $q = -p$ and

$$g_0^4 \delta^{ab} \int_p \frac{e^{ip(x-z)}}{p^2} e^{-(t+s)p^2} \int_0^t du \int_\ell \frac{1}{\ell^2} e^{-2u\ell^2} (-D+1) C_A \delta_{\mu\nu}. \tag{2.27}$$

Now, in the dimensional regularization,

$$\int_\ell \frac{1}{\ell^2} e^{-2u\ell^2} = \frac{1}{(4\pi)^{D/2}} \frac{1}{D/2-1} (2u)^{-D/2+1}, \tag{2.28}$$

and (recall $\int_\ell \frac{1}{\ell^2} = 0$!) note that

$$\int_0^t du\, u^{-D/2+1} = \frac{1}{-D/2+2} t^{-D/2+2} \sim \frac{1}{\epsilon} + \ln t. \tag{2.29}$$

Thus, the flow time integral can produce the pole singularity $1/\epsilon$. As the result,

$$(\text{4th diagram})|_{\text{div.}} = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \frac{g_0^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{1}{2} \ln(4\pi t) + \frac{1}{2} \ln(4\pi s) \right] (-3) C_A \delta_{\mu\nu}, \tag{2.30}$$

where we have added the contribution of the diagram obtained by the exchange, $(t, x, a) \leftrightarrow (s, y, b)$.

The above example illiterates the general situation: For a fixed generic value of the flow times, all the loop integral are absolutely convergent because of the Gaussian damping factors. When the flow time are integrated, however, the Gaussian factors can becomes unity depending on the value of the flow times and the integrals generally exhibit UV divergences.

The second diagram yields, after now-familiar manipulations,

$$\begin{aligned} & g_0^4 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \int_0^t du \int_\ell \frac{1}{\ell^2 (\ell-p)^2} e^{-2u\ell^2} e^{2u\ell\cdot p} \\ & \quad \times C_A \left[2(D-2)\ell_\mu \ell_\nu - (D-3)\ell_\mu p_\nu - 4p_\mu \ell_\nu + 2\delta_{\mu\nu} \ell \cdot (\ell+p) \right]. \end{aligned} \tag{2.31}$$

It turns out that we can set $p = 0$ and $D = 4$ to find the divergent part. Hence,

$$\begin{aligned} & g_0^4 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \int_0^t du \int_\ell \frac{1}{(\ell^2)^2} e^{-2u\ell^2} C_A \left(4\ell_\mu \ell_\nu + 2\delta_{\mu\nu} \ell^2 \right) \\ & = g_0^4 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \int_0^t du \int_\ell \frac{1}{\ell^2} e^{-2u\ell^2} 3 C_A \delta_{\mu\nu}, \end{aligned} \tag{2.32}$$

and

$$(\text{2nd diagram})|_{\text{div.}} = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \frac{g_0^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{1}{2} \ln(4\pi t) + \frac{1}{2} \ln(4\pi s) \right] 3 C_A \delta_{\mu\nu}. \tag{2.33}$$

A similar but somewhat more tedious calculation shows

$$
\begin{aligned}
&\text{(3rd diagram)}\\
&= g_0^4\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}e^{-(t+s)p^2}\int_0^t du\int_0^u dv\int_\ell \frac{1}{\ell^2}e^{-2u\ell^2}e^{-(u-v)2\ell\cdot p}\\
&\qquad\times C_A\bigl\{2\left(\ell^2\delta_{\mu\nu}-2\ell_\mu\ell_\nu+2p_\mu\ell_\nu-\ell\cdot p\delta_{\mu\nu}\right)\\
&\qquad\qquad+(2\ell_\mu+p_\mu)\left[(2D-1)\ell_\nu-p_\nu\right]\\
&\qquad\qquad+2\left[\ell_\mu(\ell+p)_\nu-2(\ell+p)_\mu\ell_\nu+2p\cdot(\ell+p)\delta_{\mu\nu}-p_\mu(\ell+p)_\nu\right]\bigr\}.
\end{aligned}
\tag{2.34}
$$

In the most singular term, we can neglect the external momentum and

$$
\text{(3rd diagram)}|_{\text{div.}} = g_0^4\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}e^{-(t+s)p^2}\int_0^t du\int_0^u dv\int_\ell \frac{1}{\ell^2}e^{-2u\ell^2}C_A 2(\ell^2\delta_{\mu\nu}+4\ell_\mu\ell_\nu). \tag{2.35}
$$

Hence,

$$
\text{(3rd diagram)}|_{\text{div.}} = g_0^4\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}e^{-(t+s)p^2}\int_0^t du\int_0^u dv\int_\ell e^{-2u\ell^2}4C_A\delta_{\mu\nu}. \tag{2.36}
$$

Since

$$
\int_\ell e^{-2u\ell^2} = \frac{1}{(4\pi)^{D/2}}(2u)^{-D/2}, \tag{2.37}
$$

$$
\int_0^t du\int_0^u dv\int_\ell e^{-2u\ell^2} \sim \frac{1}{(4\pi)^2}\frac{1}{4}\left[\frac{1}{\epsilon}+\ln(4\pi t)\right], \tag{2.38}
$$

and, adding the contribution of the diagram with the exchange, $(t,x,a)\leftrightarrow(s,y,b)$, we have

$$
\text{(3rd diagram)}|_{\text{div.}} = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}e^{-(t+s)p^2}\frac{g_0^2}{(4\pi)^2}\left[\frac{1}{\epsilon}+\frac{1}{2}\ln(4\pi t)+\frac{1}{2}\ln(4\pi s)\right]2C_A\delta_{\mu\nu}. \tag{2.39}
$$

Finally, the first diagram is

$$
\begin{aligned}
&\frac{1}{2}g_0^4C_A\delta^{ab}\int_p e^{ip(x-y)}e^{-(t+s)p^2}\int_0^t du\int_0^s dv\int_\ell \frac{1}{\ell^2(\ell+p)^2}e^{-2(u+v)\ell^2}e^{-(u+v)2\ell\cdot p}\\
&\quad\times\left[2\delta_{\mu\rho}\ell_\sigma+2\delta_{\mu\sigma}(\ell+p)_\rho+\delta_{\rho\sigma}(2\ell+p)_\mu\right]\left[2\delta_{\nu\rho}\ell_\sigma+2\delta_{\nu\sigma}(\ell+p)_\rho+\delta_{\rho\sigma}(2\ell+p)_\nu\right].
\end{aligned}
\tag{2.40}
$$

The most singular part is

$$
\frac{1}{2}g_0^4C_A\delta^{ab}\int_p e^{ip(x-y)}e^{-(t+s)p^2}\int_0^t du\int_0^s dv\int_\ell \frac{1}{(\ell^2)^2}e^{-2(u+v)\ell^2}\left[8\delta_{\mu\nu}\ell^2+(24+4D)\ell_\mu\ell_\nu\right]. \tag{2.41}
$$

However, since

$$
\int_0^t du\int_0^s dv\int_\ell \frac{1}{\ell^2}e^{-2(u+v)\ell^2} \overset{D\to 4}{\to} \frac{1}{(4\pi)^2}\frac{1}{2}\left[t\ln\left(1+\frac{s}{t}\right)+s\ln\left(1+\frac{t}{s}\right)\right], \tag{2.42}
$$

this diagram is finite.

In total, the sum of flow line diagrams (for $\lambda_0=\alpha_0=1$) is

$$
g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{p^2}e^{-(t+s)p^2}\frac{g_0^2}{(4\pi)^2}\left[\frac{1}{\epsilon}+\frac{1}{2}\ln(4\pi t)+\frac{1}{2}\ln(4\pi s)\right]2C_A\delta_{\mu\nu}. \tag{2.43}
$$

This precisely cancels the divergence from the diagrams consisting only from the Yang–Mills vertices. As the result, recalling $g_0^2 = \mu^{2\epsilon} g^2 + \cdots$, to the one-loop,

$$\begin{aligned}&\left\langle B_\mu^a(t,x) B_\nu^b(t,y)\right\rangle \\ &= g^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2} e^{-(t+s)p^2} \left\{1 + \frac{g^2}{(4\pi)^2} C_A \left[\ln(4\pi\mu^2 t) + \ln(4\pi\mu^2 s)\right]\right\} \delta_{\mu\nu} p^2 + (\text{finite}).\end{aligned} \tag{2.44}$$

This shows some remarkable facts: First, the two-point function of the flowed gauge field becomes UV finite only after the renormalization of parameters, g_0 and λ_0. *No wave function renormalization is required.* This is remarkable because the flowed field B is a certain (albeit very complicated) combination of the *bare* gauge field A through the flow equation. The latter requires the wave function renormalization, but the former does not. Second, after the parameter renormalization, the limit $t = s \to 0$ can be nontrivial as the above expression illustrates. It does *not* simply reduce to $\langle A_\mu^a(x) A_\nu^b(y)\rangle$ that is UV diverging (unless the wave function renormalization). In a sense, the flow and the renormalization do not commute. Since $\langle A_\mu^a(x) A_\nu^b(y)\rangle$ is UV diverging, it depends on the regularization adopted. On the other hand, $\langle B_\mu^a(t,x) B_\nu^b(t,y)\rangle$ is UV finite and can have a universal meaning that is independent of the regularization. This feature of the gradient flow is a key to obtain a universal representation of the energy–momentum tensor by employing the gradient flow.

2.2.3 General cases

The general statement that we will prove later is that any correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n)\rangle, \qquad t_1 > 0, \ldots, t_n > 0, \tag{2.45}$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.

If this statement is true, we see that the correlation function is finite even if some spacetime points collide, say $x_1 \to x_2$.

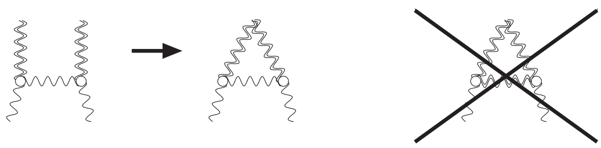


Fig. 2.7

The new loop always contains the gaussian damping factor $\sim e^{-tp^2}$ which makes integral finite; no new UV divergences arise.

Any composite operators of the flowed gauge field $B_\mu(t,x)$ are automatically renormalized UV finite quantities, although the flowed field is a certain combination of the bare gauge field. Such UV finite quantities must be independent of the regularization.

Chapter 3

Proof of the renormalizability of the gradient flow

Chapter 4

Energy–momentum tensor in gauge theory

4.1 One-loop coefficients by the background field method